MATH 211
Calculus II Integral Calculus

Module 2 Applications of Definite Integrals
Topic 3: Arc Length

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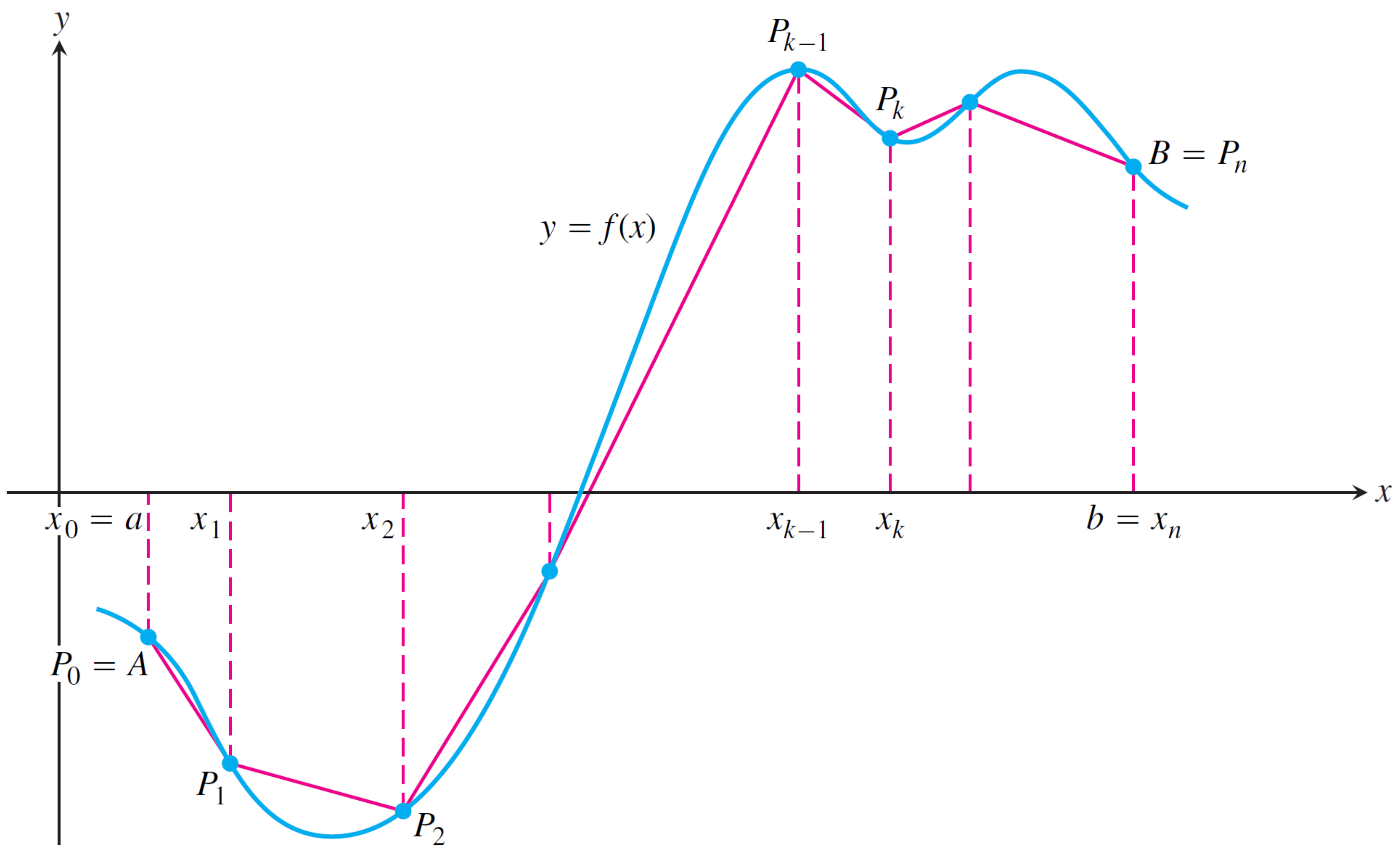
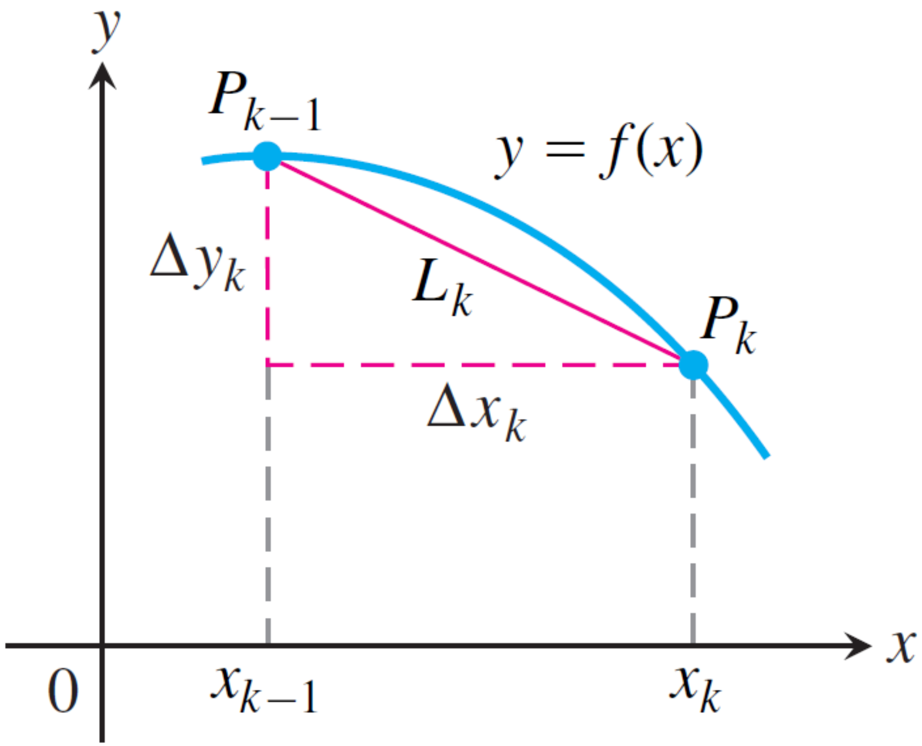
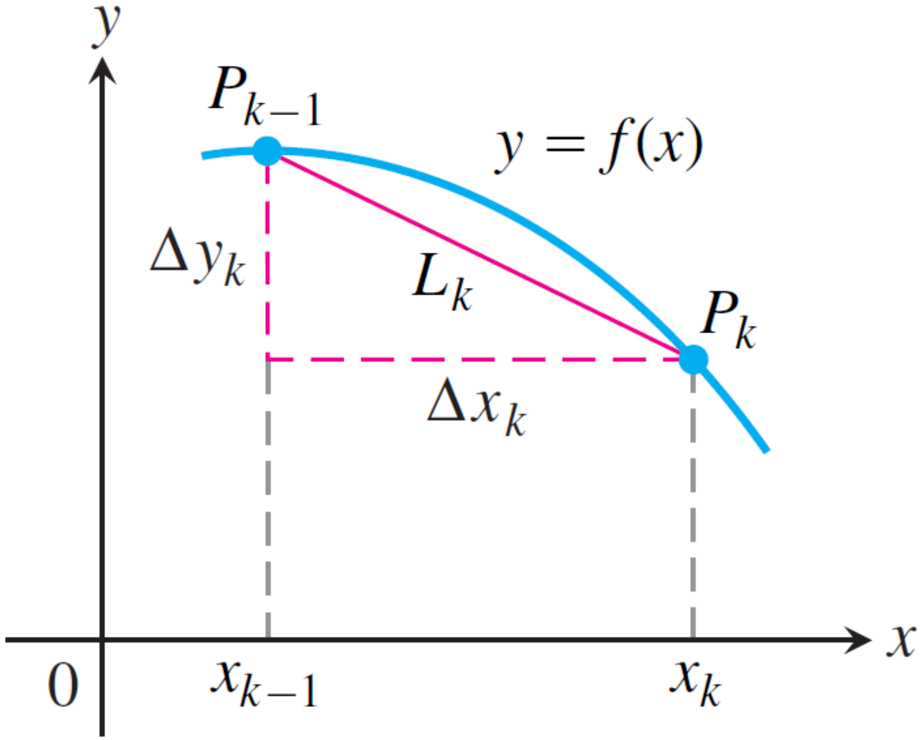
Piecewise Linear: Do the Zigzag!



On Each Linear Segment:

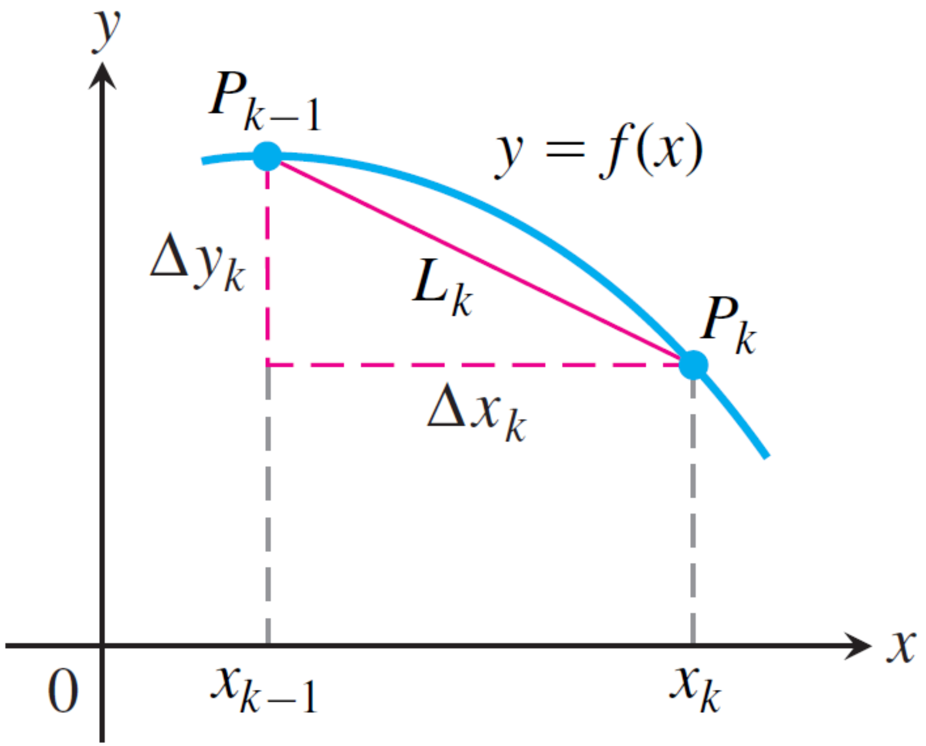


On Each Linear Segment:



$$
\begin{aligned}
L_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\
&= \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2} \\
&= \sqrt{1 + (f'(c_k))^2}\ \Delta x_k
\end{aligned}
$$

On Each Linear Segment:



$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$
$$= \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2}$$
$$= \sqrt{1 + (f'(c_k))^2}\ \Delta x_k$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\, \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

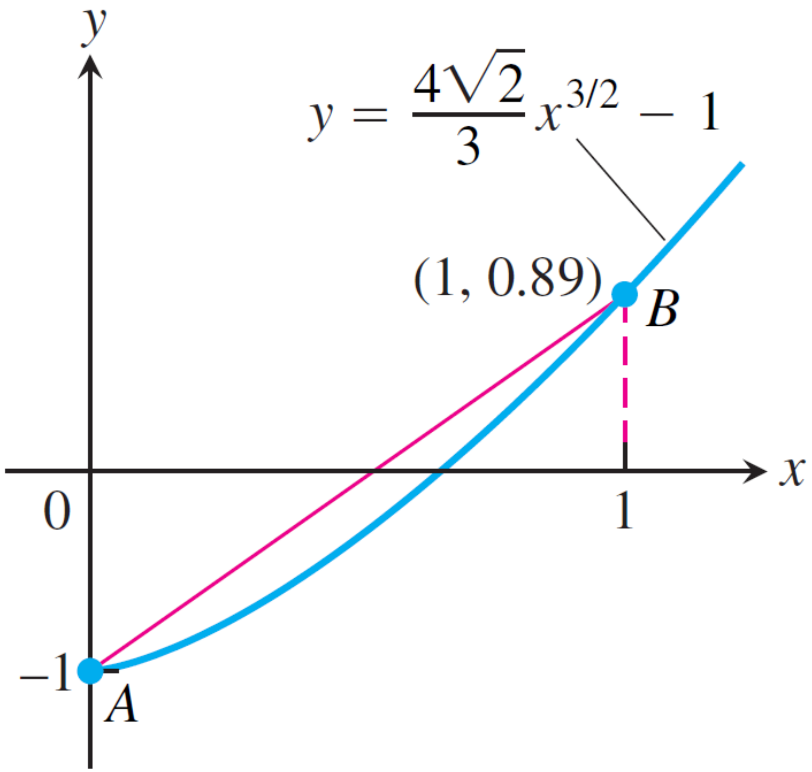
The Arc Length Formula

DEFINITION If f' is continuous on $[a, b]$, then the **length** (**arc length**) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (3)$$

EXAMPLE 1 Find the length of the curve (Figure 6.24)

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$



Check your Understanding

Find the length of the curve

$$y = \frac{1}{2}(e^x + e^{-x}), \qquad 0 \leq x \leq 2.$$

The Arc Length Function $s(x)$

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

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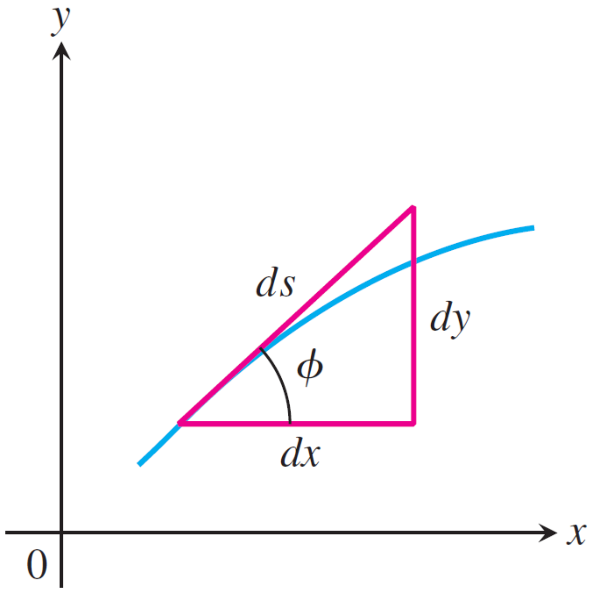
$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

$$ds = \sqrt{dx^2 + dy^2}$$



Example

Find the arc length function for the curve

$$f(x) = \frac{4\sqrt{2}}{3}x^{3/2} - 1, x \geq 0$$

taking $A = (0, -1)$ as the beginning point.