# MATH 215 DISCRETE MATHEMATICS INSTRUCTOR: P. WENG

Counting and Probability
Suggested Problems

## Basic Counting Skills, Inclusion-Exclusion, and Complement

1. (a) An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?
   (b) A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?
2. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?
3. (a) How many different three-letter initials can people have?
   (b) How many different three-letter initials with none of the letters repeated can people have?
4. (a) How many bit strings of length ten both begin and end with a 1?
   (b) How many bit strings are there of length six or less, not counting the empty string?
   (c) How many bit strings of length $n$, where $n$ is a positive integer, start and end with $1$s?
5. How many strings are there of four lowercase letters that have the letter x in them?
6. (a) How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?
   (b) How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?
7. (a) How many subsets of a set with 100 elements have more than one element?
   (b) A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length $n$ are palindromes?
8. (a) Use the principle of inclusion-exclusion to find the number of positive integers less than $1,000,000$ that are not divisible by either $4$ or by $6$.
   (b) A wired equivalent privacy (WEP) key for a wireless fidelity (WiFi) network is a string of either 10, 26, or 58 hexadecimal digits. How many different WEP keys are there? (There are 16 place values for hexadecimal numbers: $0$ to $9$, $A$, $B$, $C$, $D$, $E$, and $F$.)

## Permutations and Subsets

9. Let $S = \{1, 2, 3, 4, 5\}$.
   (a) List all the 3-permutations of $S$.
   (b) List all the 3-subsets of $S$.
10. Find the value of each of these quantities.
   (a) $P(6, 2)$
   (b) $P(6, 4)$
   (c) $P(8, 0)$

(d) $P(10, 10)$

11. Find the value of each of these quantities.
(a) $\binom{6}{2}$ (b) $\binom{6}{4}$ (c) $\binom{8}{0}$ (d) $\binom{10}{10}$

12. In how many different orders can five runners finish a race if no ties are allowed?

13. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

14. How many bit strings of length $12$ contain
    (a) exactly three 1s?
    (b) at most three 1s?
    (c) at least three 1s?
    (d) an equal number of 0s and 1s?

15. In how many ways can a set of two positive integers less than 100 be chosen?

16. Suppose that there are $9$ faculty members in the mathematics department and $11$ in the computer science department. How many ways are there to form a basketball team that has 3 math professors and 2 computer science professors?

17. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
    (a) are there in total?
    (b) contain exactly three heads?
    (c) contain at least three heads?
    (d) contain the same number of heads and tails?

18. Thirteen people on a softball team show up for a game.
    (a) How many ways are there to choose 10 players to take the field?
    (b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
    (c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

19. A club has 25 members.
    (a) How many ways are there to choose four members of the club to serve on an executive committee?
    (b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

20. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

## Pigeonhole Principle

21. (a) Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.
    (b) What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

22. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
    (a) How many balls must she select to be sure of having at least three balls of the same color?
    (b) How many balls must she select to be sure of having at least three blue balls?

23. Assuming that no one has more than $1,000,000$ hairs on the head of any person and that the population of New York City was $8,008,278$ in 2010, show there had to be at least nine people in New York City in 2010 with the same number of hairs on their heads.

## Uniform Discrete Distributions

24. (a) What is the probability that a card selected at random from a standard deck of 52 cards is an ace?
    (b) What is the probability that a fair die comes up six when it is rolled?
    (c) What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?
    (d) What is the probability that a randomly selected day of a leap year (with 366 possible days) is in April?

25. (a) What is the probability that a five-card poker hand contains the ace of hearts?
    (b) What is the probability that a five-card poker hand does not contain the queen of hearts?
    (c) What is the probability that a five-card poker hand contains the two of diamonds and the three of spades?
    (d) What is the probability that a five-card poker hand contains the two of diamonds, the three of spades, the six of hearts, the ten of clubs, and the king of hearts?
    (e) What is the probability that a five-card poker hand contains exactly one ace?

26. (a) What is the probability that a fair die never comes up an even number when it is rolled six times?
    (b) What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

27. In a super lottery, players win a fortune if they choose the eight numbers selected by a computer from the positive integers not exceeding 100. What is the probability that a player wins this super lottery?

## Distributions and Probability of Unions and Complements

28. Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.

29. What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?

30. A pair of dice is loaded. The probability that a 4 appears on the first die is $2/7$, and the probability that a 3 appears on the second die is $2/7$. Other outcomes for each die appear with probability $1/7$. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

31. Assume that the year has 366 days and all birthdays are equally likely.

    (a) Find the smallest number of people you need to choose at random so that the probability that at least one of them has a birthday today exceeds $1/2$.

    (b) Find the smallest number of people you need to choose at random so that the probability that everyone has a distinct birthday is below $1/2$.

32. Assume that the probability a child is a boy is $0.51$ and that the sexes of children born into a family are independent. What is the probability that a family of five children has...

| | |
|---|---|
| (a) exactly three boys? | (b) at least one boy? |
| (c) at least one girl? | (d) all children of the same sex? |

## Conditional Probability and Bayes' Theorem

33. A wedding party of eight people is lined up in a random order.

    (a) What is the probability that the bride is next to the groom?

    (b) What is the probability that maid of honor is in the leftmost position?

    (c) Determine whether the two events are independent. Give your reasoning.

34. A red die and a blue die are thrown. Define the following events:
    $A$: **The sum is even.** $B$: **The sum is at least 10.** $C$: **The red die comes up 5.**
    Find the following probabilities.

| | | |
|---|---|---|
| (a) $Pr(A)$, | (b) $Pr(B)$, | (c) $Pr(C)$ |
| (d) $Pr(A\|C)$ | (e) $Pr(B\|C)$, | (f) $Pr(A\|B)$. |

35. Sally has two coins. The first coin is a fair coin and the second coin is biased. The biased coin comes up heads with probability .75 and tails with probability .25. She selects a coin at random and flips the coin ten times. Out of the ten coin flips, 7 flips come up heads and 3 come up tails. What is the probability that she selected the biased coin?

36. Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 14 of the time it lands on six, and 16 of the time it lands on each of 2, 3, 4 and 5, and 112 of the time on 1. You choose a die at random, and spin it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die?

37. Assume one person out of 10,000 is infected with HIV, and there is a test in which 2.5% of all people test positive for the virus although they do not really have it. If you test negative on this test, then you definitely do not have HIV. Let $H$ be the event of having HIV and $T$ be the event of testing positive. Find the following.

    (a) $\Pr(T|H)$, the probability of testing positive *for someone with HIV*.

(b) $\Pr(H \cap T)$, the probability of having HIV and testing positive.

(c) $\Pr(T|\overline{H})$, the probability of testing positive *for someone without HIV*.

(d) $\Pr(\overline{H} \cap T)$, the probability of *not* having HIV and testing positive.

(e) $\Pr(T)$, the probability of testing positive.

(f) $\Pr(H|T)$, the probability of having HIV *for someone who tests positive.*

38. In Small Town MN, 10% of residents are teenagers. 95% of teenagers use Facebook, while only 60% of the rest of the town use Facebook. Let $T$ be the set of teenagers and $F$ be the set of Facebook users.

    (a) Find $\Pr(F \cap T)$, the probability that a resident is a teenager and a Facebook user.

    (b) Find $\Pr(F \cap \overline{T})$, the probability that a resident is a non-teenager and a Facebook user.

    (c) Find $\Pr(F)$, the probability that a resident is a Facebook user.

    (d) Find $\Pr(T|F)$, the probability for a Facebook user to be a teenager.

## Solutions

1. (a) $27 \cdot 37$ (b) $12 \cdot 2 \cdot 3$
2. $4 \cdot 6$
3. (a) $26^3$ (b) $26 \cdot 25 \cdot 24$
4. (a) $2^8$ (b) $2 + 2^2 + \cdots + 2^6$ (c) $1$ string when $n = 1$; $2^{n-2}$ strings when $n \geq 2$.
5. $26^4 - 25^4$
6. (a) $2 \cdot 26^3 \cdot 10^3$ (b) $26^3 \cdot 10^3 + 26^4 \cdot 10^2$
7. (a) $2^{100} - 1 - 10$ (b) $2^{n/2}$ when $n$ is even; $2^{(n+1)/2}$ when $n$ is odd.
8. (a) $999999 - \left(\lfloor \frac{999999}{4} \rfloor + \lfloor \frac{999999}{6} \rfloor - \lfloor \frac{999999}{12} \rfloor\right)$ (b) $16^{10} + 16^{28} + 16^{58}$
9. Skip
10. (a) $6 \cdot 5$ (b) $6 \cdot 5 \cdot 4 \cdot 3$ (c) $1$ (d) $10 \cdot 9 \cdot \cdots \cdot 1 = 10!$
11. (a) $15$ (b) $15$ (c) $1$ (d) $1$
12. $P(5,5) = 5!$
13. $P(6,6) = 6!$
14. (a) $\binom{12}{3}$ (b) $\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3}$
    (c) $2^{12} - \binom{12}{0} - \binom{12}{1} - \binom{12}{2}$ (d) $\binom{12}{6}$
15. $\binom{99}{2}$
16. $\binom{9}{3} \cdot \binom{11}{2}$
17. (a) $2^8$ (b) $\binom{8}{3}$ (c) $2^8 - \binom{8}{0} - \binom{8}{1} - \binom{8}{2}$ (d) $\binom{8}{4}$
18. (a) $\binom{13}{10}$ (b) $P(13,10)$ (c) $\binom{13}{10} - \binom{10}{10}$
19. (a) $\binom{25}{4}$ (b) $P(25,4)$
20. $\binom{10}{3} \cdot \binom{15}{3}$
21. (a) $\lceil \frac{30}{26} \rceil = 2$ (b) $50 \cdot 99 + 1$
22. (a) $2 \cdot 2 + 1 = 5$ (b) $10 + 3 = 13$
23. $\lceil \frac{8008278}{1000000} \rceil = 9$.
24. (a) $\frac{4}{52}$ (b) $\frac{1}{6}$ (c) $\frac{50}{100}$ (d) $\frac{30}{366}$

25. (a) $\dfrac{\binom{51}{4}}{\binom{52}{5}}$ (b) $\dfrac{\binom{51}{5}}{\binom{52}{5}}$ (c) $\dfrac{\binom{50}{3}}{\binom{52}{5}}$ (d) $\dfrac{1}{\binom{52}{5}}$ (e) $\dfrac{\binom{4}{1}\cdot\binom{48}{4}}{\binom{52}{5}}$

26. (a) $\left(\frac{1}{2}\right)^6$ (b) $\frac{33}{100}$

27. $\dfrac{1}{\binom{100}{8}}$

28. $P(2) = P(4) = 0.3$, $P(1) = P(3) = P(5) = P(6) = 0.1$

29. $P(Head) = 3/4$, $P(Tail) = 1/4$.

30. $\frac{9}{49}$

31. (a) Solve $n$ for $1 - \left(\frac{365}{366}\right)^n > \frac{1}{2}$
(b) Solve $n$ for $\left(\frac{366}{366}\right) \cdot \left(\frac{365}{366}\right) \cdot \cdots \cdot \left(\frac{366-(n-1)}{366}\right) < \frac{1}{2}$.

32. (a) $0.51^3$ (b) $1 - 0.49^3$ (c) $1 - 0.51^3$ (d) $0.51^3 + 0.49^3$

33. (a) $\frac{P(2,2)\cdot P(7,7)}{P(8,8)} = \frac{1}{4}$ (b) $\frac{1}{8}$
(c) No. $P$(bride-next-to-groom $\wedge$ maid-of-honor-in-leftmost) $= \frac{1\cdot P(2,2)\cdot P(6,6)}{P(8,8)} = \frac{1}{28} \neq \frac{1}{4} \cdot \frac{1}{8}$.

34. (a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{6}$ (d) $\frac{1}{2}$ (e) $\frac{1}{3}$ (f) $\frac{2}{3}$

35. $\dfrac{\frac{1}{10{,}000}\cdot 1}{\frac{1}{10{,}000}\cdot 1 + \frac{9{,}999}{10{,}000}\cdot 2.5\%}$