Suggested Problems for *Logic and Proof*

The following problems are from *Discrete Mathematics and Its Applications* by Kenneth H. Rosen.

1. Which of these are propositions? What are the truth values of those that are propositions?
   (a) Stop or I will shoot.
   (b) What time is it?
   (c) There are no black flies in Maine.
   (d) $4 + x = 5$.
   (e) The moon is made of green cheese.
   (f) $2^n \geq 100$.

2. What is the negation of each of these propositions?
   (a) Jennifer and Teja are friends.
   (b) There are 13 items in a baker's dozen.
   (c) Abby sent more than 100 text messages every day.
   (d) 121 is a perfect square.

3. Let p and q be the propositions "The election is decided" and "The votes have been counted", respectively. Express each of these compound propositions as an English sentence.
   (a) $\neg p$
   (b) $p \vee q$
   (c) $\neg p \wedge q$
   (d) $q \rightarrow p$
   (e) $\neg q \rightarrow \neg p$
   (f) $\neg p \rightarrow \neg q$
   (g) $p \leftrightarrow q$

4. Let p, q, and r be the propositions

   p : Grizzly bears have been seen in the area.

   q : Hiking is safe on the trail.

   r : Berries are ripe along the trail.

   Write these propositions using p, q, and r and logical connectives (including negations).
   (a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.
   (b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.
   (c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.
   (d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.

(e) For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.

(f) Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

5. Determine whether each of these conditional statements is true or false.

   (a) If $1 + 1 = 2$, then $2 + 2 = 5$.

   (b) If $1 + 1 = 3$, then $2 + 2 = 4$.

   (c) If $1 + 1 = 3$, then $2 + 2 = 5$.

   (d) If monkeys can fly, then $1 + 1 = 3$.

6. Write each of these statements in the form "if p, then q" in English. [Hint: Refer to the list of common ways to express conditional statements provided in this section.]

   (a) It is necessary to wash the boss's car to get promoted.

   (b) Winds from the south imply a spring thaw.

   (c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.

   (d) Willy gets caught whenever he cheats.

   (e) You can access the website only if you pay a subscription fee.

   (f) Getting elected follows from knowing the right people.

   (g) Carol gets seasick whenever she is on a boat.

7. Construct a truth table for each of these compound propositions.

   (a) $p \oplus p$

   (b) $p \oplus \neg p$

   (c) $p \oplus \neg q$

   (d) $p \rightarrow \neg q$

   (e) $(p \vee q) \rightarrow (p \wedge q)$

   (f) $(p \vee \neg q) \leftrightarrow (p \rightarrow q)$

8. Construct a truth table for each of these compound propositions.

   (a) $p \rightarrow (\neg q \vee r)$

   (b) $\neg p \rightarrow (q \rightarrow r)$

   (c) $p \vee (q \wedge r)$

   (d) $(p \rightarrow q) \wedge (\neg p \rightarrow r)$

   (e) $(p \leftrightarrow \neg r) \wedge (\neg q \leftrightarrow r)$

9. When three professors are seated in a restaurant, the hostess asks them: "Does everyone want coffee?" The first professor says: "I do not know." The second professor then says: "I do not know." Finally, the third professor says: "No, not everyonewants coffee." The hostess comes back and gives coffee to the professors who want it. How did she figure out who wanted coffee?

10. (Optional). On the island of knights and knaves created by Smullyan, knights always tell the truth and knaves always lie. You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions?

(a) A says ”At least one of us is a knave” and B says nothing.
(b) A says ”The two of us are both knights” and B says ”A is a knave.”
(c) A says ”I am a knave or B is a knight” and B says nothing.
(d) Both A and B say ”I am a knight.”
(e) A says ”We are both knaves” and B says nothing.

11. (Optional). On an island there are three kinds of people: knights who always tell the truth, knaves who always lie, and spies who can either lie or tell the truth. You encounter three people, A, B, and C. You know one of these people is a knight, one is a knave, and one is a spy. Each of the three people knows the type of person each of other two is. For each of these situations, if possible, determine whether there is a unique solution and determine who the knave, knight, and spy are. When there is no unique solution, list all possible solutions or state that there are no solutions.

    (a) A says ”C is the knave,” B says, ”A is the knight,” and C says ”I am the spy.”
    (b) A says ”I am the knight,” B says ”I am the knave,” and C says ”B is the knight.”
    (c) A says ”I am the knave,” B says ”I am the knave,” and C says ”I am the knave.”
    (d) A says ”I am the knight,” B says ”A is telling the truth,” and C says ”I am the spy.”
    (e) A says ”I am the knight,” B says, ”A is not the knave,” and C says ”B is not the knave.”
    (f) A says ”I am the knight,” B says ”I am the knight,” and C says ”I am the knight.”
    (g) A says ”I am not the spy,” B says ”I am not the spy,” and C says ”A is the spy.”
    (h) A says ”I am not the spy,” B says ”I am not the spy,” and C says ”I am not the spy.”

12. Use a truth table to verify the following laws.

    (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
    (b) $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$
    (c) $(p \rightarrow q) \equiv (\neg p \vee q)$
    (d) $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$
    (e) $\neg(p \oplus q) \equiv (p \leftrightarrow q)$
    (f) $\neg(p \rightarrow q) \equiv (p \wedge \neg q)$

13. Show that each of these conditional statements is a tautology by using truth tables.

    (a) $[\neg p \wedge (p \vee q)] \rightarrow q$
    (b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

14. Show that $(p \rightarrow q) \rightarrow r$ is NOT equivalent to $p \rightarrow (q \rightarrow r)$.

15. Let $P(x)$ be the statement ”the word x contains the letter a.” What are these truth values?

    (a) $P(\text{orange})$
    (b) $P(\text{lemon})$
    (c) $P(\text{true})$
    (d) $P(\text{false})$

16. Let $N(x)$ be the statement ”x has visited North Dakota,” where the domain consists of the students in your school. Express each of these quantifications in English.

(a) $\exists x\, N(x)$
(b) $\forall x\, N(x)$
(c) $\neg\exists x\, N(x)$
(d) $\exists x\, \neg N(x)$
(e) $\neg\forall x\, N(x)$
(f) $\forall x\, \neg N(x)$

17. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

(a) $\forall x\, (R(x) \to H(x))$
(b) $\forall x\, (R(x) \wedge H(x))$
(c) $\exists x\, (R(x) \to H(x))$
(d) $\exists x\, (R(x) \wedge H(x))$

18. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

(a) $P(0)$
(b) $P(1)$
(c) $P(2)$
(d) $P(-1)$
(e) $\exists x\, P(x)$
(f) $\forall x\, P(x)$

19. Let $Q(x)$ be the statement "$x+1 > 2x$." If the domain consists of all integers, what are these truth values?

(a) $Q(0)$
(b) $Q(-1)$
(c) $Q(1)$
(d) $\exists x\, Q(x)$
(e) $\forall x\, Q(x)$
(f) $\exists x\, \neg Q(x)$
(g) $\forall x\, \neg Q(x)$

20. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

(a) $\exists x\, P(x)$
(b) $\forall x\, P(x)$
(c) $\neg\exists x\, P(x)$
(d) $\neg\forall x\, P(x)$
(e) $\forall x\, ((x \neq 3) \to P(x)) \wedge \exists x\, \neg P(x)$

21. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

    (a) All dogs have fleas.
    (b) There is a horse that can add.
    (c) Every koala can climb.
    (d) No monkey can speak French.
    (e) There exists a pig that can swim and catch fish.

22. Determine whether $\forall x\,(P(x) \rightarrow Q(x))$ and $(\forall x\, P(x)) \rightarrow (\forall x\, Q(x))$ are logically equivalent. Justify your answer.

23. Determine whether $\exists x\,(P(x) \vee Q(x))$ and $(\exists x\, P(x)) \vee (\exists x\, Q(x))$ are logically equivalent. Justify your answer.

24. Let $P(x, y)$ be the statement "Student x has taken class y," where the domain for x consists of all students in your class and the domain for y consists of all computer science courses at Metro State University. Express each of these quantifications in English.

    (a) $\exists x \exists y\, P(x, y)$
    (b) $\exists x \forall y\, P(x, y)$
    (c) $\forall x \exists y\, P(x, y)$
    (d) $\exists y \forall x\, P(x, y)$
    (e) $\forall y \exists x\, P(x, y)$
    (f) $\forall x \forall y\, P(x, y)$

25. Let $Q(x, y)$ be the statement "student x has been a contestant on quiz show y." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at Metro State University and the domain for y consists of all quiz shows on television.

    (a) There is a student at your school who has been a contestant on a television quiz show.
    (b) No student at your school has ever been a contestant on a television quiz show.
    (c) There is a student at your school who has been a contestant on Jeopardy and on Wheel of Fortune.
    (d) Every television quiz show has had a student from your school as a contestant.
    (e) At least two students from your school have been contestants on Jeopardy.

26. Translate each of these quantifications into English and determine its truth value.

    (a) $\forall x \in \mathbf{R}\,(x^2 \neq -1)$
    (b) $\exists x \in \mathbf{Z}\,(x^2 = 2)$
    (c) $\forall x \in \mathbf{Z}\,(x^2 > 0)$
    (d) $\exists x \in \mathbf{Z}\,(x^2 = x)$
    (e) $\exists x \in \mathbf{R}\,(x^3 = -1)$
    (f) $\exists x \in \mathbf{Z}\,(x + 1 > x)$
    (g) $\forall x \in \mathbf{Z}\,(x - 1 \in \mathbf{Z})$

(h) $\forall x \in \mathbf{Z}\,(x^2 \in \mathbf{Z})$

27. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

    (a) The product of two negative integers is positive.
    (b) The average of two positive integers is positive.
    (c) The difference of two negative integers is not necessarily negative.
    (d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

28. Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?

    (a) $Q(1, 1)$
    (b) $Q(2, 0)$
    (c) $\forall y\, Q(1, y)$
    (d) $\exists x\, Q(x, 2)$
    (e) $\exists x \exists y\, Q(x, y)$
    (f) $\forall x \exists y\, Q(x, y)$
    (g) $\exists y \forall x\, Q(x, y)$
    (h) $\forall y \exists x\, Q(x, y)$
    (i) $\forall x \forall y\, Q(x, y)$

29. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

    (a) $\forall n \exists m\, (n^2 < m)$
    (b) $\exists n \forall m\, (n < m^2)$
    (c) $\forall n \exists m\, (n + m = 0)$
    (d) $\exists n \forall m\, (nm = m)$
    (e) $\exists n \exists m\, (n^2 + m^2 = 5)$
    (f) $\exists n \exists m\, (n^2 + m^2 = 6)$
    (g) $\exists n \exists m\, (n + m = 4 \wedge n - m = 1)$
    (h) $\exists n \exists m\, (n + m = 4 \wedge n - m = 2)$
    (i) $\forall n \forall m \exists p\, (p = (m + n)/2)$

30. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

    (a) $\forall x \exists y\, (x^2 = y)$
    (b) $\forall x \exists y\, (x = y^2)$
    (c) $\exists x \forall y\, (xy = 0)$
    (d) $\exists x \exists y\, (x + y \neq y + x)$
    (e) $\forall x (x \neq 0 \rightarrow \exists y\, (xy = 1))$
    (f) $\exists x \forall y\, (y \neq 0 \rightarrow xy = 1)$

(g) $\forall x \exists y\,(x + y = 1)$

(h) $\exists x \exists y\,(x + 2y = 2 \wedge 2x + 4y = 5)$

(i) $\forall x \exists y\,(x + y = 2 \wedge 2x - y = 1)$

(j) $\forall x \forall y \exists z\,(z = (x + y)/2)$

31. Determine the truth value of the statement $\exists x \forall y\,(x \leq y^2)$ if the domain for the variables consists of

    (a) the positive real numbers.

    (b) the integers.

    (c) the nonzero real numbers.

32. Use direct proof and show the following.

    (a) The square of an even number is an even number.

    (b) The additive inverse, or negative, of an even number is an even number.

    (c) If $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.

    (d) The product of two odd numbers is odd.

    (e) The product of two rational numbers is rational.

    (f) If x is rational and $x \neq 0$, then $1/x$ is rational.

    (g) Assume n is an integer. If n is an odd number, then so is n^2.

    (h) Assume n is an integer. If n is an even number, then so is n^2.

33. Prove the following using proof by contrapositive.

    (a) If x is irrational, then $1/x$ is irrational.

    (b) Assume n is an integer. If n^2 is an even number, then so is n.

    (c) Assume n is an integer. If n^2 is an odd number, then so is n.

34. Prove the following using proof by contradiction.

    (a) If x is irrational, then $1/x$ is irrational.

    (b) Assume n is an integer. If $n^2 + 8$ is an odd number, then so is n.

    (c) Assume n is an integer. If $5n + 2$ is an even number, then so is n.

35. Prove or disprove.

    (a) The product of two irrational numbers is irrational.

    (b) Assume m and n are integers. If mn is even, then m is even or n is even.

    (c) Assume m and n are real numbers. If mn is irrational, then m is irrational or n is irrational.

    (d) The sum of an irrational number and a rational number is irrational.

Solutions to Selected Problems

32. (c) *Proof.* Assume $m+n=2k$ and $n+p=2h$, $k,\ h\in\mathbf{Z}$. Then $m=2k-n$, $p=2h-n$, so

$$m+p=(2k-n)+(2h-n)=2k+2h-2n=2(k+h-n),\ \text{with } k+h-n\in\mathbf{Z}.$$

Therefore $m+p$ is an even number. □

33. (a) The contrapositive of the proposition states that **if** $1/x$ **is rational then** x **is rational.**

Proof. Note that $\frac{1}{x}$ is always nonzero. Assume $\frac{1}{x}$ is rational, then $\frac{1}{x}=\frac{m}{n}$ for $m,\ n\in\mathbf{Z}$ and $n\neq 0$. Furthermore, $m\neq 0$ because $\frac{1}{x}\neq 0$, so

$$\frac{1}{x}=\frac{m}{n},\ x=\frac{n}{m},\ \text{where } m,\ n\in\mathbf{Z} \text{ and } m\neq 0.$$

So x is rational. □

(b) The contrapositive of the proposition states that **if** n **is odd then** n^2 **is odd.**

Proof. Assume n is odd: $n=2k+1$ for some $k\in\mathbf{Z}$. Then

$$n^2=(2k+1)^2=4k^2+4k+1=2(2k^2+2k)+1,\ \text{where } 2k^2+2k\in\mathbf{Z}.$$

So n^2 is odd. □

(c) The contrapositive of the proposition states that **if** n **is even then** n^2 **is even.**

Proof. Assume n is even: $n=2k$ for some $k\in\mathbf{Z}$. Then

$$n^2=(2k)^2=4k^2=2(2k^2),\ \text{where } 2k^2\in\mathbf{Z}.$$

So n^2 is even. □

34. (a) *Proof.* Assume x is irrational and $1/x$ is rational. Note that $1/x$ is always nonzero. Since $1/x$ is rational, $\frac{1}{x}=\frac{m}{n}$ for $m,\ n\in\mathbf{Z}$ and $n\neq 0$. Furthermore, $m\neq 0$ because $1/x\neq 0$, so

$$\frac{1}{x}=\frac{m}{n},\ x=\frac{n}{m},\ \text{where } m,\ n\in\mathbf{Z} \text{ and } m\neq 0.$$

So x is rational, which contradicts our initial assumption that it is irrational. □

(b) *Proof.* Assume n^2+8 is odd and n is even. Since n is even, $n=2k$ for some $k\in\mathbf{Z}$. It follows that

$$n^2+8=(2k)^2+8=4k^2+8=2(2k^2+4),\ \text{where } 2k^2+4\in\mathbf{Z}.$$

So n^2+8 is an even number, which contradicts our initial assumption that it is odd. □

35. (a) This proposition is false. Consider the counterexample: $\sqrt{2}$ and $2\sqrt{2}$ are both irrational, but their product $\sqrt{2} \cdot 2\sqrt{2} = 4$ is rational.

(b) This proposition is true. Prove by contradiction as follows. Assume that mn is even, and that m and n are both odd. Since m and n are both odd, $m = 2k + 1$ and $n = 2h + 1$ for some $k,\ h \in \mathbf{Z}$. Then

$$mn = (2k+1)(2h+1) = 4kh + 2k + 2h + 1 = 2(2kh + k + h) + 1, \text{ where } 2kj + k + h \in \mathbf{Z}.$$

So mn is odd,which contradicts the initial assumption that mn is even.

Remark. You can also prove this by contrapositive.

(c) This proposition is true. Prove by contrapositive as follows. The contrapositive of the proposition states that **if** m **and** n **are both rational then** mn **is rational.** Assume m and n are rational, then $m = \frac{h}{k}$ and $n = \frac{p}{q}$ for some $h,\ k,\ p,\ q \in \mathbf{Z}$ and $k,\ q \neq 0$. Then

$$mn = \frac{h}{k} \cdot \frac{p}{q} = \frac{hp}{kq}, \text{ where } hp,\ kq \in \mathbf{Z}, kq \neq 0.$$

Therefore mn is rational.

(d) This proposition is true.