Suggested Problems for *Properties of Integers*

The following problems are from *Discrete Mathematics and Its Applications* by Kenneth H. Rosen.

1. Compute the value of the following expressions:

   (a) $323 \mod 5$
   (b) $323 \text{ div } 5$
   (c) $-323 \mod 5$
   (d) $-323 \text{ div } 5$
   (e) $327 \mod 3$
   (f) $(64 \cdot (-67) + 201) \bmod 7$
   (g) $38^{12} \mod 6$
   (h) $38^{12} \mod 3$

2. Compute the value of the following expressions:

   (a) $344 \mod 5$
   (b) $344 \text{ div } 5$
   (c) $-344 \mod 5$
   (d) $-344 \text{ div } 5$
   (e) $387 \mod 3$
   (f) $(72 \cdot (-65) + 211) \bmod 7$
   (g) $(77 \cdot (-65) + 147) \bmod 7$
   (h) $(44^{12}) \bmod 6$
   (i) $17^{12} \mod 3$

3. Prove the following.

   (a) For all integers a, b and c, if $a|b$ and $b|c$, then $a|c$.
   (b) For all integers d, a and b, if $d|a$ and $d|b$, then $d|(3a - 4b)$.
   (c) For all integers d, a and b, if $d|a$ and $d|b$, then $d^2|(a^2 + b^2)$.

4. Group the following numbers according to equivalence mod 11. That is, put two numbers in the same group if they are equivalent mod 11.

$$-57,\ 17,\ 108,\ 0,\ -110,\ -93,\ 1111,\ 130,\ 232$$

5. Group the following numbers according to equivalence mod 13. That is, put two numbers in the same group if they are equivalent mod 13.

$$-63,\ 54,\ -41,\ 11,\ 13,\ 76,\ 80,\ 130,\ 132,\ 137$$

6. Give the multiplication table for $\mathbb{Z}_6$.

7. Some numbers and their prime factorizations are given below.

$$\begin{aligned} 532 &= 2^2 \cdot 7 \cdot 19 \\ 648 &= 2^3 \cdot 3^4 \\ 1083 &= 3 \cdot 19^2 \\ 15435 &= 3^2 \cdot 5 \cdot 7^3 \end{aligned}$$

   Use the prime factorizations to compute the following quantities:

   (a) $\gcd(532, 15435)$
   (b) $\gcd(648, 1083)$
   (c) $\text{lcm}(532, 1083)$
   (d) $\text{lcm}(1083, 15435)$
   (e) $\text{lcm}(648, 15435)$

8. Some numbers and their prime factorizations are given below.

$$\begin{aligned} 140 &= 2^2 \cdot 5 \cdot 7 \\ 1078 &= 2 \cdot 7^2 \cdot 11 \\ 175 &= 5^2 \cdot 7 \\ 25480 &= 2^3 \cdot 5 \cdot 7^2 \cdot 13 \end{aligned}$$

Use the prime factorizations to compute the following quantities:

(a) $\gcd(1078, 140)$
(b) $\gcd(1078, 25489)$
(c) $\mathrm{lcm}(1078, 140)$
(d) $\mathrm{lcm}(175, 25480)$
(e) $\mathrm{lcm}(140, 25480)$
(f) $175 \cdot 25480$
(g) $25480/140$

9. Let p, p_1, p_2, $\ldots$, p_k be prime numbers and α, $\alpha 1$, $\alpha 2$, $\ldots$, α be positive integers.

   (a) How many distinct divisors are there for p^α?

   (b) How many distinct divisors are there for $p_1^{\alpha 1} p_2^{\alpha 2} \cdots p_k^{\alpha_k}$?

10. If you are looking for a prime factor of $895,117$, what is the largest factor you would have to check?

11. Omitted.

12. Given two integers m and n, a *linear combination* of a and b is a number of the form

$$a \cdot m + b \cdot n$$

where a and b are integers.

   (a) **Examples**. (a) 13 is a linear combination of 2 and 3 because $13 = 2 \cdot 2 + 3 \cdot 3$. (b) 1 is a linear combination of 2 and 3 because $1 = 5 \cdot 2 + (-3) \cdot 3$.

   (b) **Exercise**: Show that 1 is a linear combination of 5 and 7.

13. Given m and n, find $\gcd(m, n)$.

(a) $m = 56$, $n = 42$
(b) $m = 81$, $n = 60$
(c) $m = 259$, $n = 77$
(d) $m = 72$, $n = 42$
(e) $m = 80$, $n = 61$
(f) $m = 630$, $n = 147$

Solutions

1. (a) 3 (b) 64 (c) 2 (d) -65 (e) 0 (f) 1 (g) 4 (h) 1

2. Omitted.

3. (a) $a|b$: $b = am$ for some integer m. $b|c$: $c = bn$ for some integer n. So $c = bn = (am)n = a(mn)$, hence $a|c$.

   (b) $d|a$ and $d|b$: $a = dm$ and $b = dn$ for integers m and n. Then $3a - 4b = 3(dm) - 4(dn) = d(3m - 4n)$, so $d|(3a - 4b)$.

   (c) $d|a$ and $d|b$: $a = dm$ and $b = dn$ for integers m and n. Then $a^2 + b^2 = (dm)^2 + (dn)^2 = d^2(m^2 + n^2)$, so $d|(a^2 + b^2)$.

4. $\{-57, 108, 130\}$, $\{17, -93\}$, $\{0, -110, 1111\}$ and $\{232\}$.

5. Omitted.

6. Use $0, 1, 2, 3, 4, 5$. For example, $5 \cdot 4 = 20 = 20 \bmod 6 = 2$, etc.

7. (a) 7 (b) 3 (c) $2^2 \cdot 3 \cdot 7 \cdot 19^2$ (d) $3^2 \cdot 5 \cdot 7^3 \cdot 19^2$, etc.

8. Skip.

9. (a) $1 + \alpha$ (b) $(1 + \alpha_1)(1 + \alpha_2) \cdots (1 + \alpha_k)$

10. $\lfloor \sqrt{895117} \rfloor = 946$.

11. Omitted.

12. (b) $1 = (3) \cdot 5 + (-2) \cdot 7$.

13. (a) 14 (b) 3 (c) 7 (d) 6 (e) 1 (f) 21