MATH 215 DISCRETE MATHEMATICS SUMMER 2015

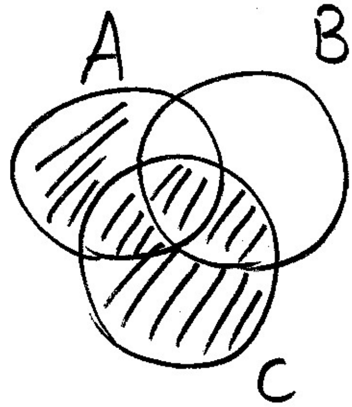
Exam I
June 9, 2015

Name: Weng

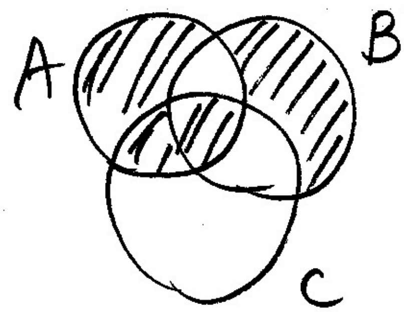
This is a 3-hour exam. You may use a calculator and a letter-sized double-sided sheet of notes. No books, cellphones or other electronic devices are allowed. You must show all your work to receive full credit for each problem you solve. The highest score you may receive is 100 points.

1. Graph the Venn diagram for the following sets.

   (a) (3 points) $(A - B) \cup C$.



   (b) (3 points) $A \oplus (B - C)$.



2. Consider the set $A = \{1, 2, 3, \emptyset, \{x, y\}\}$. Determine if each of the following propositions is true or false.

   (a) (2 points) $\{x, y\} \subset A$. TRUE FALSE

   (b) (2 points) $\emptyset \in A$ TRUE FALSE

   (c) (2 points) $\{2, 3\} \subset A$ TRUE FALSE

   (d) (2 points) $\{x\} \subset A$ TRUE FALSE

page total:

3. (7 points) Let $A_i = [0, i] = \{x : 0 \leq x \leq i\}$. Find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$.

$$\bigcup_{i=1}^{n} A_i = [0,1] \cup [0,2] \cup \cdots \cup [0,n] = [0,n] \, (= A_n).$$

$$\bigcap_{i=1}^{n} A_i = [0,1] \cap [0,2] \cap \cdots \cap [0,n] = [0,1] \, (= A_1).$$

4. (7 points) $f(x) = \dfrac{2x-1}{x+1}$. Find $f^{-1}(5)$.

$$\frac{2x-1}{x+1} = 5 \; : \qquad 2x-1 = 5(x+1)$$

$$2x-1 = 5x+5$$

$$-6 = 3x$$

$$\Rightarrow \boxed{x = -2}$$

5. (7 points) Let $A = \{a, c, e\}$ and $B = \{1, 3\}$. Find $A \times B$ and $B \times A$.

$A \times B = \{(a,1), (a,3), (c,1), (c,3), (e,1), (e,3)\}$

$B \times A = \{(1,a), (1,c), (1,e), (3,a), (3,c), (3,e)\}$

page total:

6. Answer the following questions about functions. Explain your answer.

(a) (3 points) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{x}$. Is f a function? **YES NO**

$f(x)$ is undefined for $x < 0$.

(b) (3 points) $f : \mathbf{Z} \to \mathbf{Z}$, $f(n) = |n - 3|$. Is f *one-to-one*? **YES NO**

$f(4) = f(2) = 1$

(c) (3 points) $f : \mathbf{Z} \to \mathbf{Z}$, $f(x) = x + 6$. Is f *onto*? **YES NO**

$y = x+6 \Rightarrow x = y-6$ So $f(y-6) = y \quad \forall y \in \mathbb{Z}$.

(d) (3 points) Find a function that is one-to-one and onto. Do not use the identity function $f(x) = x$. Identify the domain and target.

$f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$.

(e) (3 points) Find a function that is one-to-one but **NOT** onto. Identify the domain and target.

$f : \mathbb{N} \to \mathbb{N}$, $f(x) = x+1$.

(f) (3 points) Find a function that is onto but **NOT** one-to-one. Identify the domain and target.

$f : \mathbb{Z} \to \mathbb{N}$ $\quad f(x) = |x| + 1$.

page total:

7. (7 points) Construct a truth table for the proposition $(p \oplus \neg q) \wedge (p \to r)$.

p	q	r	$(p \oplus \neg q)$	$\wedge$	$(p \to r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	F	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

8. (7 points) Use a truth table and determine if the propositions $p \oplus (\neg q)$ and $p \leftrightarrow q$ are equivalent.

p	q		$p \oplus (\neg q)$			$p \leftrightarrow q$	
T	T	T	T	F	T	T	T
T	F	T	F	T	T	F	F
F	T	F	F	F	F	F	T
F	F	F	T	T	F	T	F

Yes, they are equivalent.

page total:

9. Let $Q(x, y)$ be the statement "$y = x^2$" for all real numbers x and y. Find the truth values for the following. **Prove your answer.**

(a) (3 points) $Q(3, 9)$ TRUE FALSE

$9 = 3^2$: True

(b) (3 points) $Q(-3, -9)$ TRUE FALSE

$-9 = (-3)^2$: False

(c) (3 points) $\forall x \exists y\, Q(x, y)$. TRUE FALSE

$\forall x$ there is a # y so that $y = x^2$.

(d) (3 points) $\exists y \forall x\, Q(x, y)$. TRUE FALSE

$\exists y$ so that for all x, $y = x^2$.

(This means y is the square of all numbers.

There doesn't exist such a number.)

(e) (3 points) $\forall y \exists x\, Q(x, y)$. TRUE FALSE

For any y, there is an x so that $y = x^2$.

This is not true. If y is negative,

$y < 0$, y is not the square of any real #.

page total:

10. (7 points) Is it true that for all real numbers x and y, $\lceil x-y \rceil = \lceil x \rceil - \lceil y \rceil$? Prove your answer.

This is false. Consider counterexample:

$x = 1$ & $y = 0.5$

$\cancel{\lceil x - y \rceil}$ $\lceil x-y \rceil = \lceil 1-0.5 \rceil = \lceil 0.5 \rceil = 1.$

$\lceil x \rceil - \lceil y \rceil = \lceil 1 \rceil - \lceil 0.5 \rceil = 1-1 = 0.$ } unequal.

$\blacksquare$

11. (7 points) Prove that the product of two rational numbers is rational.

Let $x \in \mathbb{Q}$: $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$

$y \in \mathbb{Q}$: $y = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$

$x + y = \frac{p}{q} + \frac{m}{n} = \frac{np + mq}{nq}$ where $np + mq$, $nq \in \mathbb{Z}$
$nq \neq 0$.

So $x + y \in \mathbb{Q}$.

$\blacksquare$

page total:

12. (7 points) Prove that if $2x$ is irrational then x is irrational.

Prove the contrapositive: if x is rational then $2x$ is rational.

<u>Proof</u>. Assume $x \in \mathbb{Q}$, $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$

Then $2x = 2 \cdot \frac{p}{q} = \frac{2p}{q}$ where $2p, q \in \mathbb{Z}$, $q \neq 0$.

So $2x \in \mathbb{Q}$. $\blacksquare$

(P.S. You can also prove by contradiction.)

13. (7 points) Prove that if n^2 is an even number, then so is n.

Proof by contrapositive: If n is odd then n^2 is odd.

<u>Proof</u>. Assume n is odd, $n = 2k+1$ for some $k \in \mathbb{Z}$.

Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$= 2(2k^2+2k)+1$, $2k^2+2k \in \mathbb{Z}$.

So n^2 is odd. $\blacksquare$

(P.S. You can also prove by contradiction.)

page total: