MATH 215 DISCRETE MATHEMATICS FALL 2015

Exam II
November 5, 2015

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This is a 3-hour exam. You may use a calculator and a letter-sized double-sided sheet of notes. No books, cellphones or other electronic devices are allowed. You must show all your work to receive full credit for each problem you solve. The highest score you may receive is 100 points.

1. (10 points) Use a truth table and determine if $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ are logically equivalent.

p	q	r	$p \wedge q$	$\to r$	$(p \to r)$	$\wedge$	$(q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

not equivalent

2. Let $P(x)$ be the statement "$x^2 > 4x - 2$" with domain $\mathbf{N}$.

(a) (5 points) Is $\exists x\ P(x)$ true or false? Explain.

True. Example. $x = 5$: $5^2 = 25 > 4 \cdot 5 - 2 = 18$.
So $p(5)$ is true.

Therefore $\exists x\, p(x)$ is true.

(b) (5 points) Is $\forall x\ P(x)$ true or false? Explain.

False. Counterexample. $x = 2$: $2^2 = 4 \not> 4 \cdot 2 - 2 = 6$.
So $p(2)$ is false.
Therefore $\forall x\, p(x)$ is false.

3. Consider for all real numbers x and y.

(a) (5 points) Is $\forall x \exists y\ (y = x^2)$ true or false? Explain.

This is true b/c for any real # x, we can always square it & call it y.

(b) (5 points) Is $\exists y \forall x\ (y = x^2)$ true or false? Explain.

This is false b/c no number y can be the square of all numbers x. For example, y cannot be the squares of 1 & 2 at the same time

4. (10 points) Prove directly that *if x and y are rational then $2x + 3y$ is rational.*

Assume x & y are rational, $x = \frac{m}{n}$ & $y = \frac{p}{q}$

$m, n, p, q \in \mathbb{Z}$, $n, q \neq 0$

Then $2x+3y = 2 \cdot \frac{m}{n} + 3 \cdot \frac{p}{q}$

$= \frac{2m}{n} + \frac{3p}{q} = \frac{2mq + 3np}{nq}$

where $2mq + 3np$, $nq \in \mathbb{Z}$

$nq \neq 0$.

So $2x+3y$ is rational.

$\boxed{}$

5. (10 points) Prove indirectly that *if $3n+5$ is even then n is odd.*

Contrapositive: If n is even then $3n+5$ is odd.

Proof. Assume n is even, $n = 2m$, $m \in \mathbb{Z}$.

Then $3n+5 = 3\cdot 2m + 5$

$= 6m+5$

$= 6m+4+1$

$= 2(3m+2)+1$

where $3m+2 \in \mathbb{Z}$

So, $3n+5$ is odd. $\blacksquare$

6. (a) (5 points) Find the first 5 terms for the given sequences.

$$a_0 = a_1 = 1,\ a_n = a_{n-1} + 2a_{n-2},\ n \geq 2.$$

$a_2 = a_1 + 2a_0 = 1 + 2\cdot 1 = 3$

$a_3 = 3 + 2\cdot 1 = 5$

$a_4 = 5 + 2\cdot 3 = 11$

$a_5 = 11 + 2\cdot 5 = 21$

(b) (5 points) Evaluate the summation

$$\sum_{i=5}^{8} \left\lfloor \frac{2i+5}{3} \right\rfloor.$$

$\left\lfloor \frac{2\cdot 5+5}{3} \right\rfloor + \left\lfloor \frac{2\cdot 6+5}{3} \right\rfloor + \left\lfloor \frac{2\cdot 7+5}{3} \right\rfloor + \left\lfloor \frac{2\cdot 8+5}{3} \right\rfloor$

$= \left\lfloor \frac{15}{3} \right\rfloor + \left\lfloor \frac{17}{3} \right\rfloor + \left\lfloor \frac{19}{3} \right\rfloor + \left\lfloor \frac{21}{3} \right\rfloor = 5+5+6+7 = 23$

7. (10 points) Prove that for all positive integers n,

$$\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}. \longrightarrow p(n)$$

(Step 1) $p(1)$: $\sum_{i=0}^{1} \frac{1}{2^i} = \frac{1}{2^0} + \frac{1}{2^1} = 1 + \frac{1}{2} = \frac{3}{2} = 2 - \frac{1}{2^1}$: true

(Step 2) Assume $p(k)$: $\sum_{i=0}^{k} \frac{1}{2^i} = 2 - \frac{1}{2^k}$

Consider $p(k+1)$: $\sum_{i=0}^{k+1} \frac{1}{2^i} = 2 - \frac{1}{2^{k+1}}$

$\boxed{\sum_{i=0}^{k} \frac{1}{2^i}} + \frac{1}{2^{k+1}} = 2 - \frac{1}{2^{k+1}}$

$2 - \boxed{\frac{1}{2^k}} + \frac{1}{2^{k+1}} = 2 - \frac{1}{2^{k+1}}$

$2 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} = 2 - \frac{1}{2^{k+1}}$

$2 - \frac{1}{2^{k+1}} = 2 - \frac{1}{2^{k+1}}$ $\therefore$ true

So $p(k+1)$ is true.

So $p(k) \Rightarrow p(k+1)$.

Conclusion:

By math. induction

$p(n)$ is true

for $n = 1, 2, \ldots$

8. (10 points) Let $P(n)$ be the statement that a gift of n dollars can be formed using just 5-dollar and 6-dollar gift certificates. Prove that $P(n)$ is true for $n \geq 20$.

Step 1 $p(20)$ is true: $20 = 5+5+5+5$

$p(21)$ is true: $21 = 5+5+5+6$

$p(22)$ is true: $22 = 5+5+6+6$

$p(23)$ is true: $23 = 5+6+6+6$

$p(24)$ is true: $24 = 6+6+6+6$

Step 2 Assume $p(20) \wedge p(21) \wedge \cdots \wedge p(k)$ is true, $k \geq 24$

Consider $p(k+1)$: $k+1 = 5 + (k-4)$, so the strategy for $\$(k-4)$ can be extended to form $\$(k+1)$ by adding an extra $\$5$. Since $k \geq 24$, $k-4 \geq 20$. So $p(k-4)$ is true by assumption. Therefore $p(k+1)$ is true

Conclusion. By strong induction, $p(n)$ is true for $n \geq 20$

9. (10 points) Let λ denote the empty binary string. Consider for all possible binary strings made of 0's and 1's. Give a recursive definition for the set S which is all the binary strings that don't have consecutive 0's. For example, $0101101 \in S$ and $110100 \notin S$.

$\lambda \in S$

If x ends in 0 then $x1 \in S$

If x ends in 1 then $x0 \in S$ & $x1 \in S$.

10. (10 points) Give a recursive algorithm to compute the sum of the reciprocals of the first n positive integers. The input to the algorithm is a positive integer n, the output is $\sum_{j=1}^{n} \frac{1}{j}$.

$S_1 = \frac{1}{1}$

$S_n = \frac{1}{n} + S_{n-1}$, $n = 2, 3, \ldots$

11. (Bonus Problem.)

(a) (5 points) Show that $(p \wedge q) \to r \equiv (p \wedge \neg r) \to \neg q$.

p	q	r	$(p \wedge q)$	$\to$	r	$(p \wedge \neg r)$	$\to$	$\neg q$
T	T	T	T	T	T	F	T	F
T	T	F	T	F	F	T	F	F
T	F	T	F	T	T	F	T	T
T	F	F	F	T	F	T	T	T
F	T	T	F	T	T	F	T	F
F	T	F	F	T	F	F	T	F
F	F	T	F	T	T	F	T	T
F	F	F	F	T	F	F	T	T

(b) (5 points) Use it to prove that *if x is rational and y is irrational then $x + y$ is* ***irrational.***

Rephrase: If x is rational & $x+y$ is rational then y is rational.

Proof. Assume $x = \frac{m}{n}$ & $x+y = \frac{p}{q}$. $m, n, p, q \in \mathbb{Z}$, $n, q \neq 0$

Then $y = x+y-x = \frac{p}{q} - \frac{m}{n} = \frac{np-mq}{nq}$

where $np-mq, nq \in \mathbb{Z}$, $nq \neq 0$.

So y is rational. $\blacksquare$