MATH 215 DISCRETE MATHEMATICS INSTRUCTOR: P. WENG

Relations and Directed Graphs

Suggested Problems

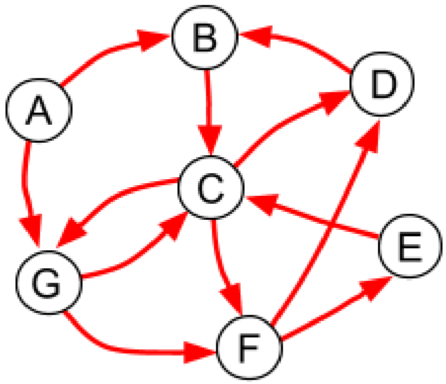
1. Draw the arrow diagram and the matrix representation for the following relation on the set $\{1,2,3,4\}$: $R = \{(1,2), (3,4), (2,3), (2,1), (3,1)\}$

2. For each of the relations below, answer the following three questions:
   Q1: Is the relation reflexive, anti-reflexive or neither?
   Q2: Is the relation symmetric, anti-symmetric or neither?
   Q3: Is the relation transitive?
   If the relation does not have one of the properties, give an example illustrating this.
   a. The domain is a set of people. (x; y) is in the relation if person x is taller than person y.
   b. The domain is a set of people. (x; y) is in the relation if person x is a first cousin of person y (i.e., a parent of person x is a sibling of a parent of person y.
   c. The domain is a set of Metro State students. (x; y) is in the relation if person x knows the student ID number for person y. You can assume that every student knows his or her own student ID number.
   d. The domain is the set of real numbers. (x; y) is in the relation if $x + y = 0$.
   e. The domain is the set of real numbers. (x; y) is in the relation if $x = 2y$.
   f. The domain is the set of real numbers. (x; y) is in the relation if $x - y$ is a rational number.
   g. The domain is the set of real numbers. (x; y) is in the relation if$x \neq y$.
   h. The domain is $A = \{a, b, c, d\}$. The relation is $\{(a, b), (a, a), (b, b), (b, a), (c, d), (d, c)\}$

3. Give an example of a relation on the set $\{1, 2, 3\}$ that is neither reflexive nor anti-reflexive.

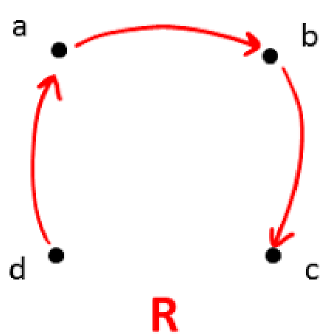
4. Is it possible to have a relation on a set that is symmetric and anti-symmetric? If so, give an example.

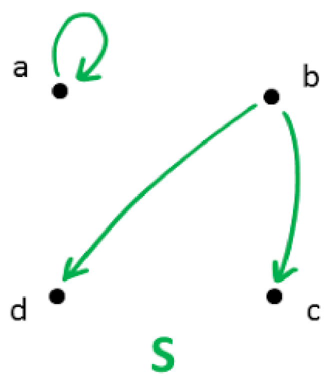
5.



a. What is the in-degree of C?
b. What is the out-degree of F?
c. What is the head of edge (E, C)?
d. What is the tail of edge (F, D)?
e. Is <G, C, F, E> a walk in the graph? Is it a path?
f. Is <G, C, D, F> a walk in the graph? Is it a path?
g. Is <G, C, D, F> a circuit in the graph? Is it a cycle?
h. Is <G, C, D, B, C, F, E, C, G> a circuit in the graph? Is it a cycle?
i. What is the longest cycle in the graph?

6.





a. Write down R.
b. Write down S.
c. Write down $S \circ R$.
d. Write down $R \circ S$.

7. Here are two relations on the set {a, b, c, d}:

$$S = \{ (a,b), (a,c), (c,d), (c,a) \}$$
$$R = \{ (b,c), (c,b), (a,d), (d,b) \}$$

Write down the set of pairs in $S \circ R$.

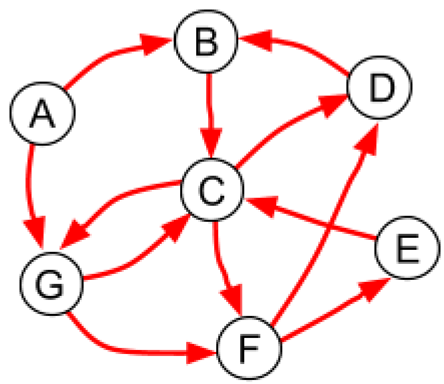
8. Define the following relations on the set R:

$$R_1 = \{(x, y): x \leq y\}$$
$$R_2 = \{(x, y): x > y\}$$
$$R_3 = \{(x, y): x < y\}$$
$$R_4 = \{(x, y): x = y\}$$

Use mathematical notation to describe the following relations:

a. $R_1 \circ R_2$
b. $R_4 \circ R_2$
c. $R_3 \circ R_4$
d. $R_3 \circ R_2$

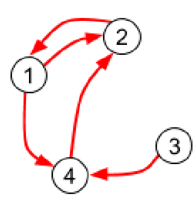
10. The diagram below shows a directed graph G:



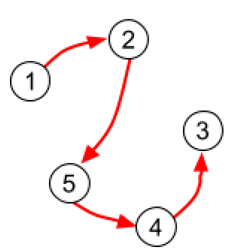
a. Is (A, B) in G^2?
b. Is (B, E) in G^3?
c. Is (G, G) in G^3?
d. Is (G, G) in G^4?
e. Is (B, B) in G^3?
f. Is (B, D) in G^5?

11. Draw a picture of G+ for each of the digraphs below:

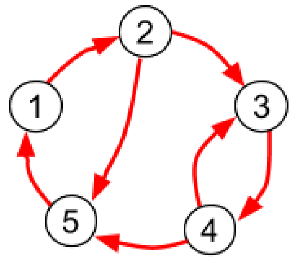
a. G_1



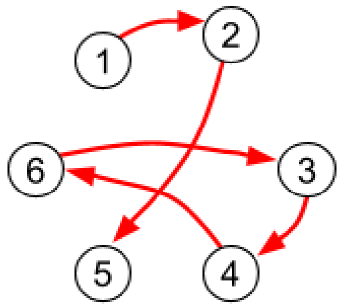
b. G_2



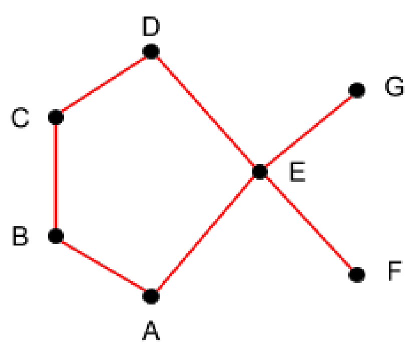
c. G_3



d. G_4



12. Consider a digraph G in which each vertex has in-degree at least one. Suppose that the relation defined by the edges of G is symmetric. Is G+ reflexive? Why or why not?
13. Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on
$$\{1, 2, 3, 4, 6, 8, 12\}.$$
14. Draw a Hasse diagram for the following relation on the set $A = \{a, b, c, d, e, f\}$.
$$R = \{(b, e), (b, d), (c, a), (c, f), (a, f), (a, a), (b, b), (c, c), (d, d), (e, e), (f, f)\}.$$
15.

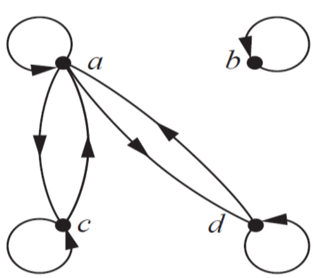


a. Name the minimal elements.
b. Name the maximal elements.
c. Which of the following pairs are comparable? (A, D), (B, E), (G, F), (D, B), (C, F), (C, E).

16. The relations below are all defined on a set of people. Which ones are equivalence relations? In the case that the relation is not an equivalence relation, give the condition that is violated.
    a. (x, y) is in the relation if person x is a first cousin of person y (i.e., a parent of person x is a sibling of a parent of person y.
    b. (x, y) is in the relation if x and y have the same birth mother.
    c. (x, y) is in the relation of x knows y's cell phone number.
17. Consider the equivalence relation on the set $S = \{7, 2, 13, 44, 56, 34, 99, 31, 4, 17\}$ such that two numbers are equivalent if they have the same remainder when divided by 5. Give the partition of S defined by the equivalence relation.
18. Let R be the set of all pairs (a, b) where a and b are mathematicians that have been co-authors on a paper.
    a. Prove whether or not R is an equivalence relation.
    b. Describe the meaning of $R \circ R$.
    c. Describe the transitive closure of R. Prove that this is an equivalence relation.
19. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.
    a. $\{(0,0), (1,1), (2,2), (3,3)\}$
    b. $\{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$
    c. $\{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$
    d. $\{(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
    e. $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$
20. Let R be the relation on the set of ordered pairs of positive integers such that ((a, b), (c, d)) ∈ R if and only if a + d = b + c. Show that R is an equivalence relation.

21—23. Determine if each of the given relations is an equivalence relation.

21.



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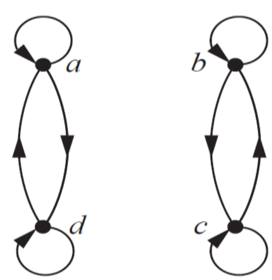
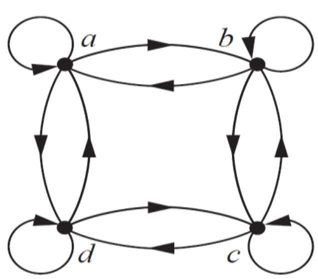


23.



Selected Solutions.

2. (a) anti-reflexive, anti-symmetric and transitive.
   (c) reflexive, neither symmetric nor anti-symmetric, and not transitive.
   (d) neither reflexive nor anti-reflexive, symmetric, and not transitive.
   (e) neither reflexive nor anti-reflexive, neither symmetric nor anti-symmetric, and not transitive.
   (g) anti-reflexive, neither symmetric nor anti-symmetric, and not transitive.

3. $R = \{(1,1)\}$

4. $\emptyset$

5. (a) 3 (b) 2 (c) C (d) F (e) yes and yes. (f) no and no. (g) no and no. (h) yes and no.

6. (a) $\{(d,a),(a,b),(b,c)\}$
   (b) $\{(a,a),(b,d),(b,c)\}$
   (c) $\{(a,d),(a,c),(d,a)\}$
   (d) $\{(a,b),(b,a)\}$

8. (a) $\{(x,y)$: x, y are any real numbers$\}$
   (b) $\{(x,y): x > y\}$
   (c) $\{(x,y): x < y\}$
   (d) $\{(x,y)$: x, y are any real numbers$\}$

10. (a) no (b)—(f) yes.

12. Yes. Since G is symmetric and each element has in-degree at least one, if $x \neq y$ and xRy then yRx. So we can always consider $x \to y \to x$, i.e. $(x,x) \in G^2$, hence it is contained in G^+.

15. (a) A and F (b) D and G (c) (A,D), (G,F) and (D,B)

16. (b) only.

17. $\{\{7,2,17\},\{13\},\{44, 34,99,4\},\{56,31\}\}$

18. (a) No because x is not his/her own co-author.

19. Only (a) and (c) are equivalence relations.

21—23. Only 22 is an equivalence relation.