

MATH 310 Multivariable Calculus

Module 6

Integration in Vector Fields

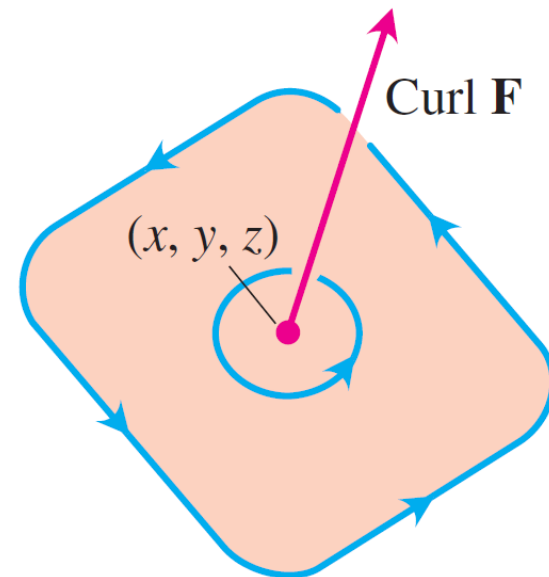


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Lesson 6G

Stoke's Theorem

π



$$\mathbf{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.$$

$$\text{curl } \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.$$

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

EXAMPLE 1 Find the curl of $\mathbf{F} = (x^2 - z)\mathbf{i} + xe^z\mathbf{j} + xy\mathbf{k}$.

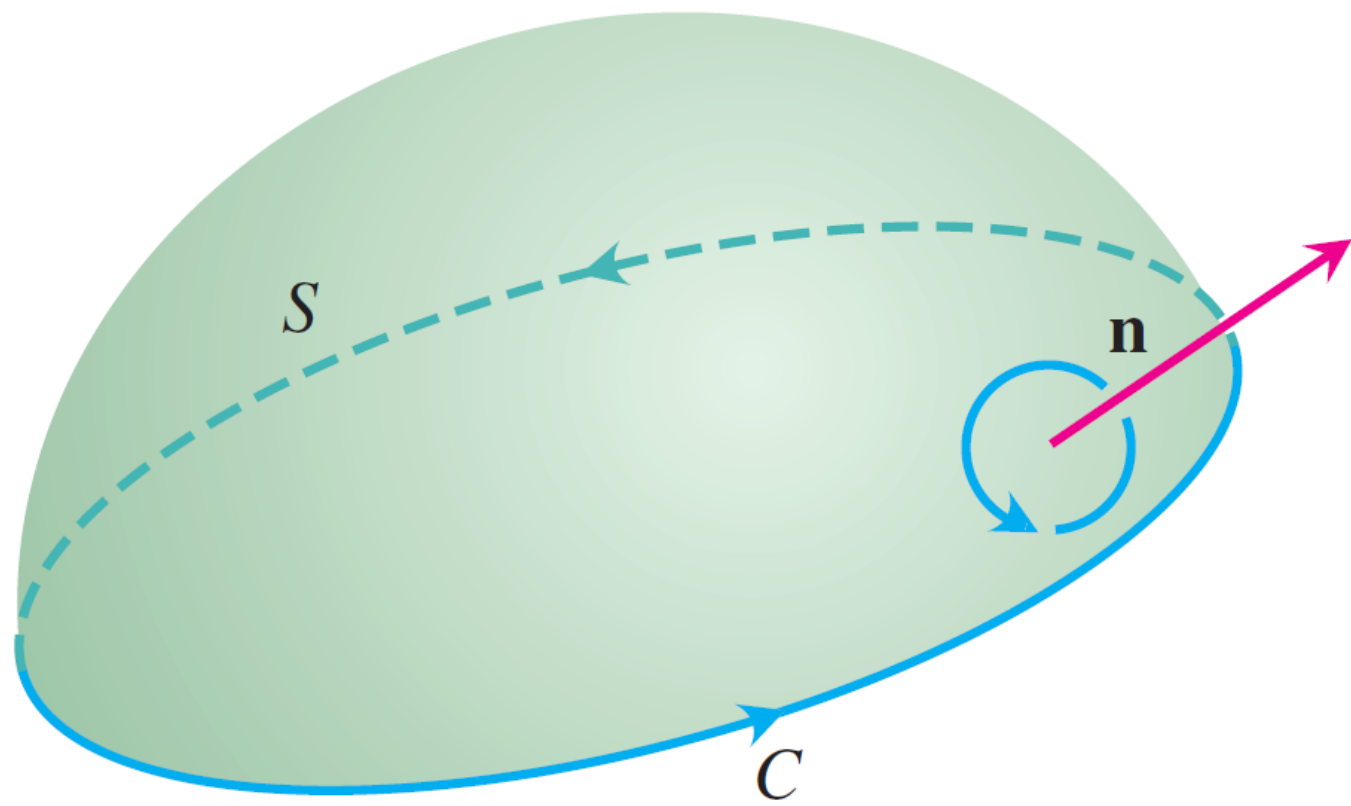
THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C . Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S . Then the circulation of \mathbf{F} around C in the direction counterclockwise with respect to the surface's unit normal vector \mathbf{n} equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S .

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma \quad (4)$$

Counterclockwise
circulation

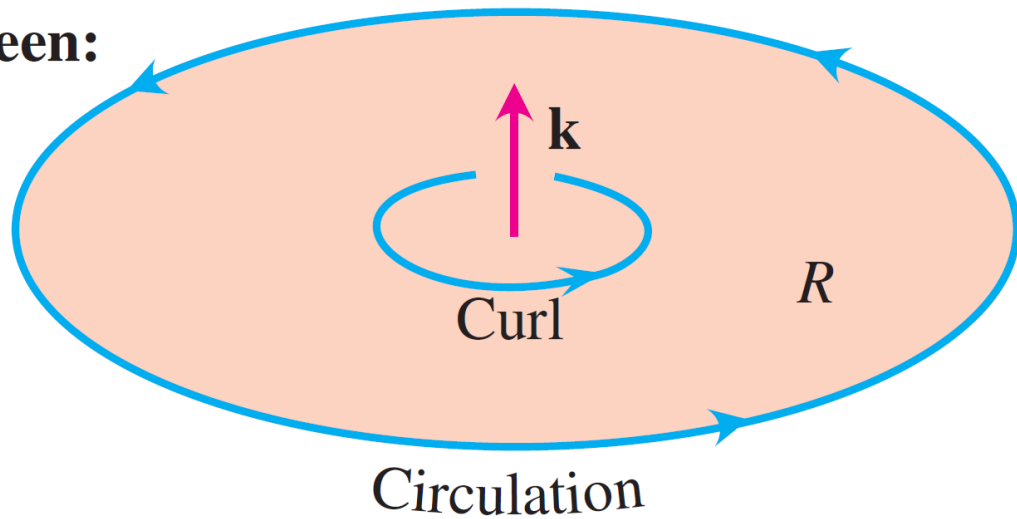
Curl integral

π

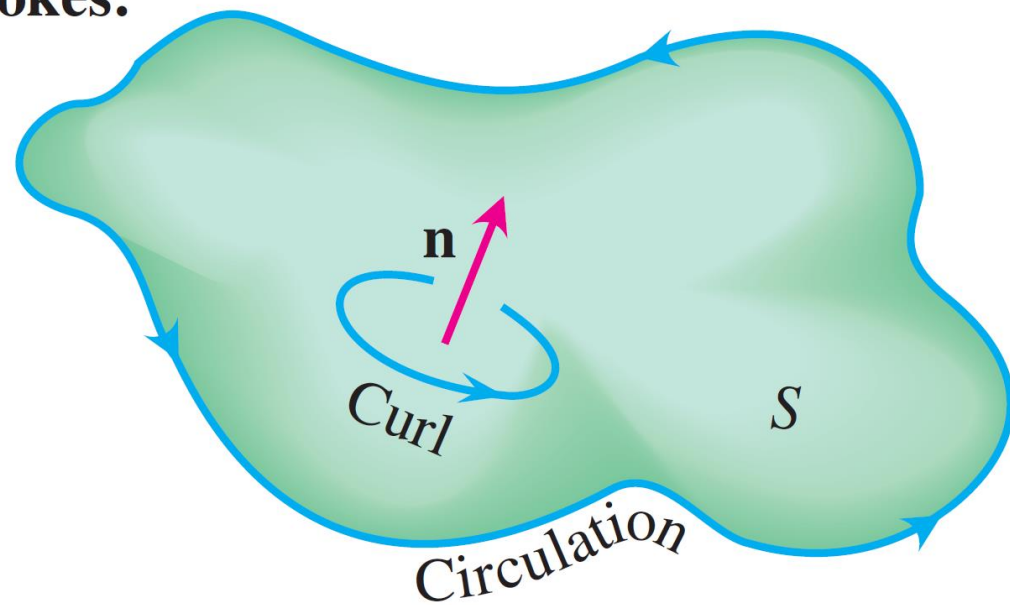


π

Green:

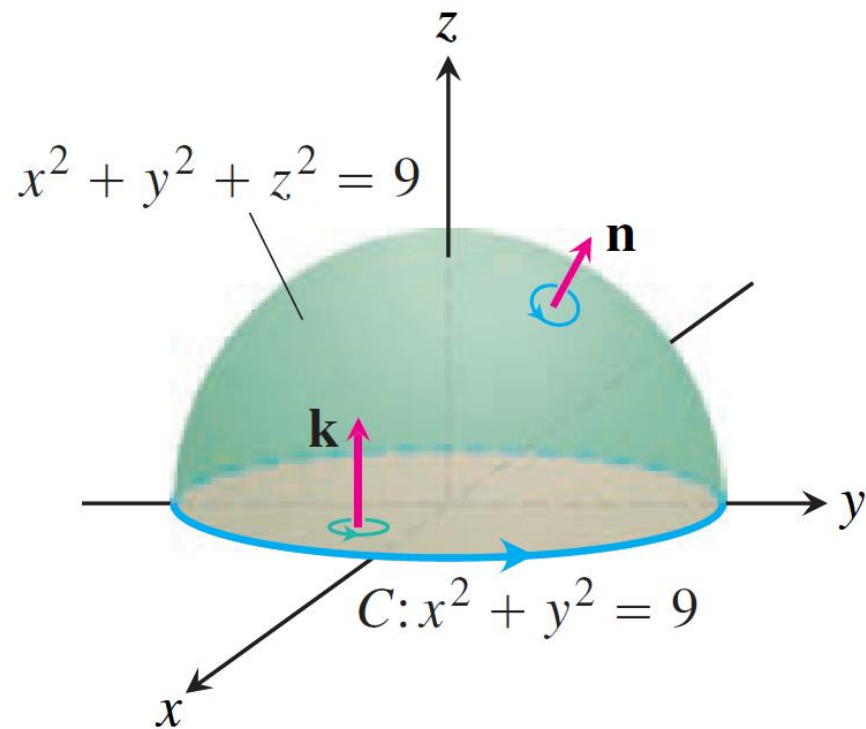


Stokes:



EXAMPLE 2 Evaluate Equation (4) for the hemisphere $S: x^2 + y^2 + z^2 = 9, z \geq 0$, its bounding circle $C: x^2 + y^2 = 9, z = 0$, and the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

$$\mathbf{r}(\theta) = (3 \cos \theta)\mathbf{i} + (3 \sin \theta)\mathbf{j}, 0 \leq \theta \leq 2\pi$$



An Important Identity

$$\text{curl grad } f = \mathbf{0} \quad \text{or} \quad \nabla \times \nabla f = \mathbf{0} \quad (8)$$

Conservative Fields and Stokes' Theorem

THEOREM 7—Curl $\mathbf{F} = \mathbf{0}$ Related to the Closed-Loop Property If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D ,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Theorem 2,
Section 16.3

\mathbf{F} conservative on D



$\mathbf{F} = \nabla f$ on D

Theorem 3,
Section 16.3



$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$
over any closed
path in D

Vector identity (Eq. 8)
(continuous second
partial derivatives)

$\nabla \times \mathbf{F} = \mathbf{0}$ throughout D

Theorem 7
Domain's simple
connectivity and
Stokes' Theorem

Lesson 6H

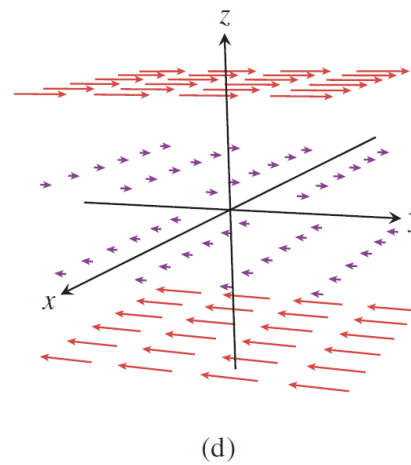
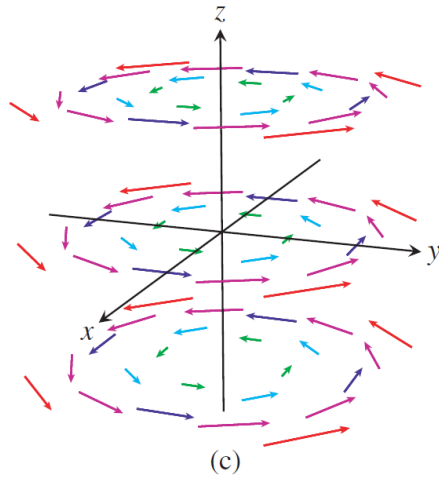
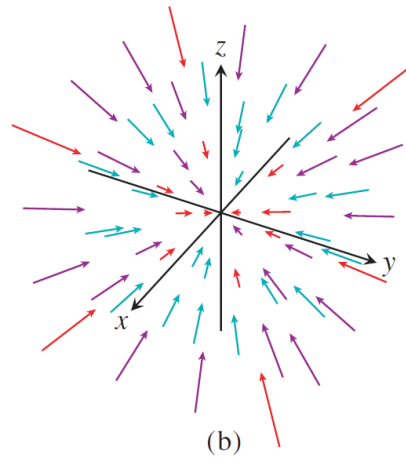
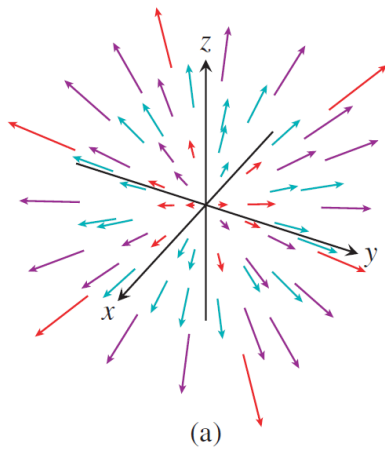
The Divergence Theorem

Divergence in Three Dimensions

The **divergence** of a vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is the scalar function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \quad (1)$$

EXAMPLE 1



(a) Expansion: $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) Compression: $\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

(c) Rotation about z-axis: $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

(d) Shearing along horizontal planes: $\mathbf{F}(x, y, z) = z\mathbf{j}$

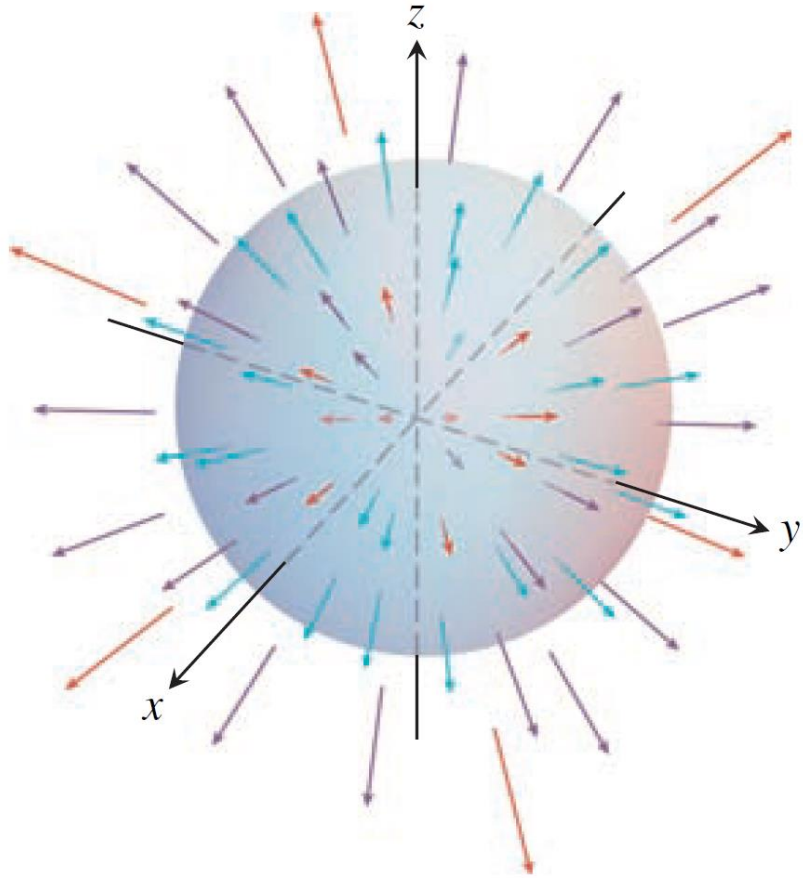
THEOREM 8—Divergence Theorem Let \mathbf{F} be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of \mathbf{F} across S in the direction of the surface's outward unit normal field \mathbf{n} equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV. \quad (2)$$

Outward
flux

Divergence
integral

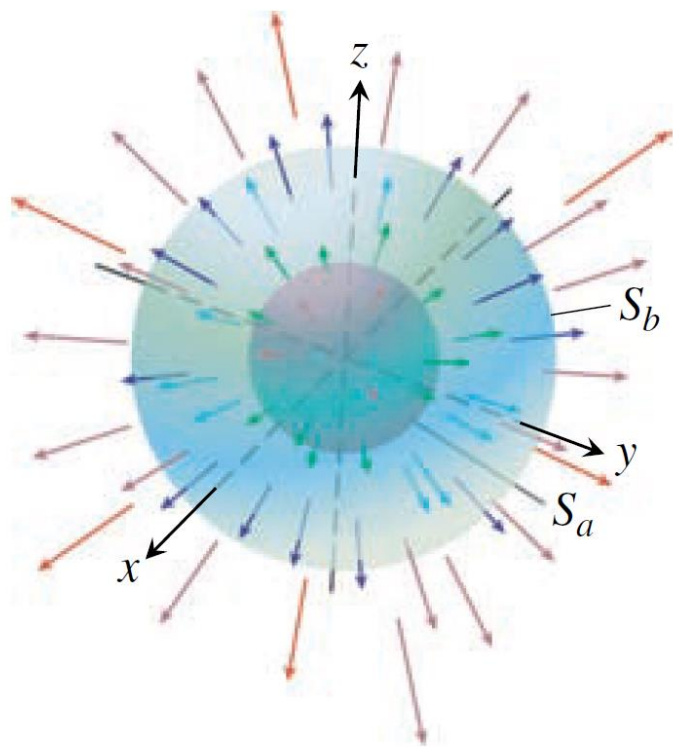
EXAMPLE 2 Evaluate both sides of Equation (2) for the expanding vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$ (Figure 16.68).



EXAMPLE 4 Find the net outward flux of the field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}, \quad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region $D: 0 < a^2 \leq x^2 + y^2 + z^2 \leq b^2$ (Figure 16.75).



Unifying the Integral Theorems

Green's Theorem and Its Generalization to Three Dimensions

Normal form of Green's Theorem:
$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \nabla \cdot \mathbf{F} \, dA$$

Divergence Theorem:
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

Tangential form of Green's Theorem:
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA$$

Stokes' Theorem:
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$