MATH 2220 Multivariable Calculus

Module 3
Multiple Integrals

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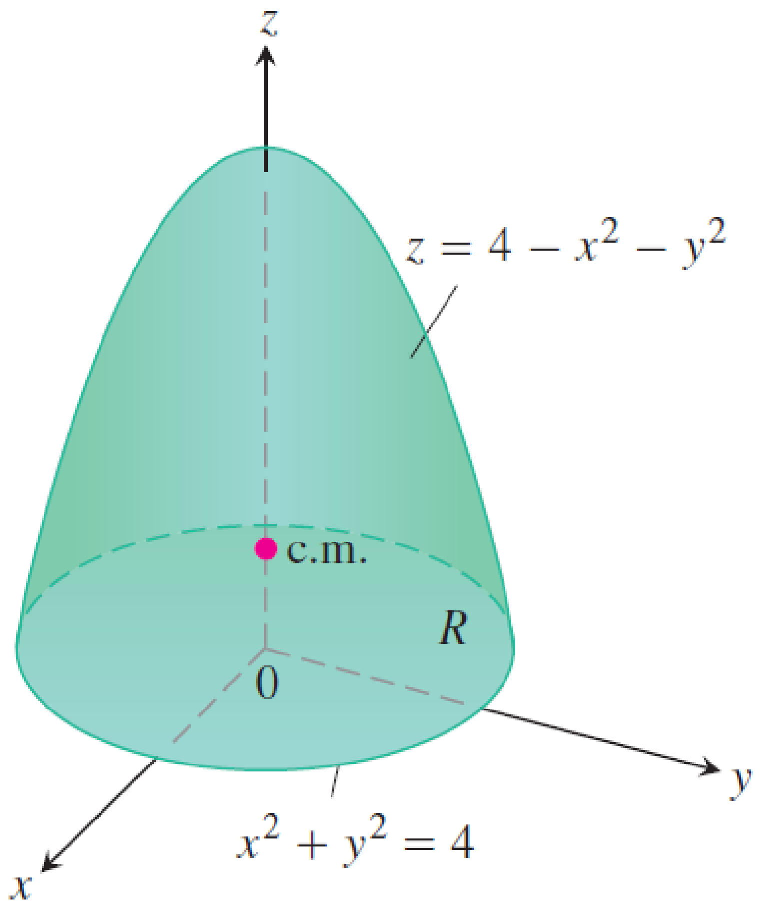
Lesson 23⅔

Center of Mass

EXAMPLE 1 Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.35).



Lesson 24

Triple Integrals in Other Coordinates

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

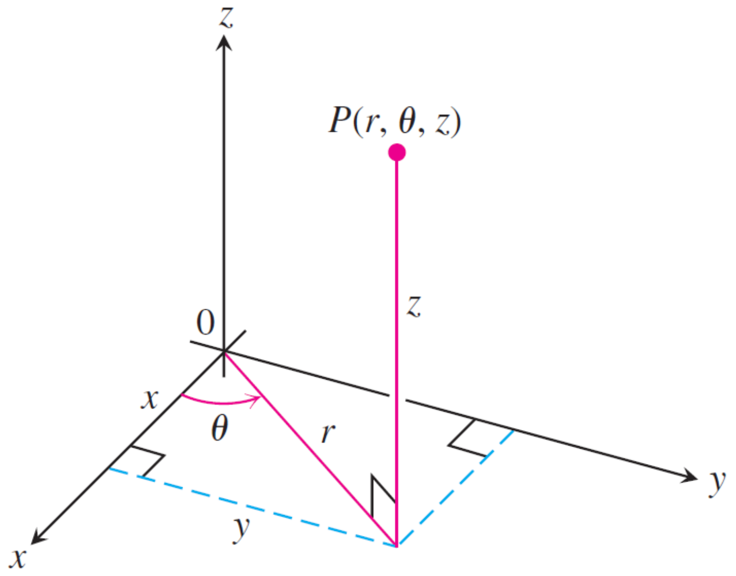
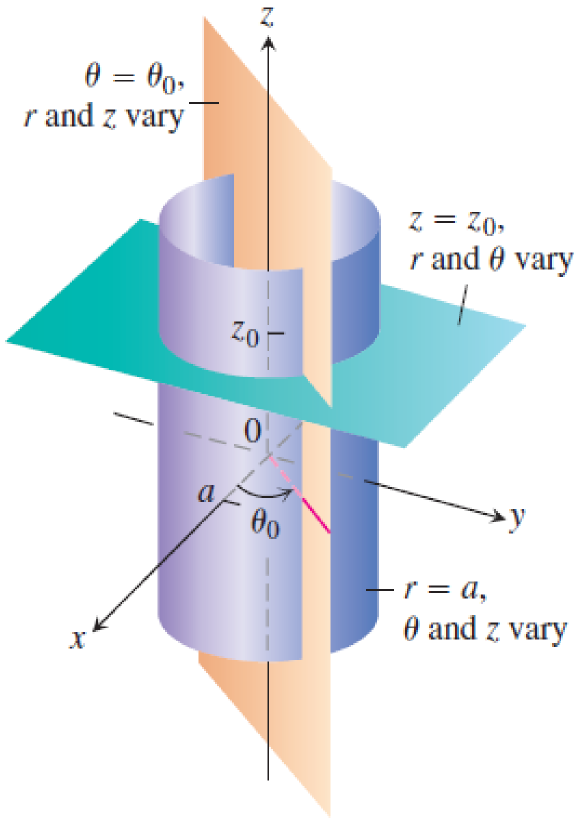


FIGURE 15.42 The cylindrical coordinates of a point in space are r, θ, and z.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$



Volume Differential in Cylindrical Coordinates

$$dV = dz\, r\, dr\, d\theta$$

$$\iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta.$$

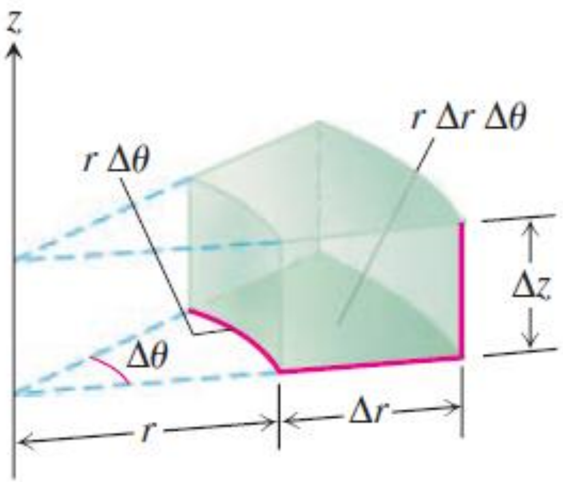
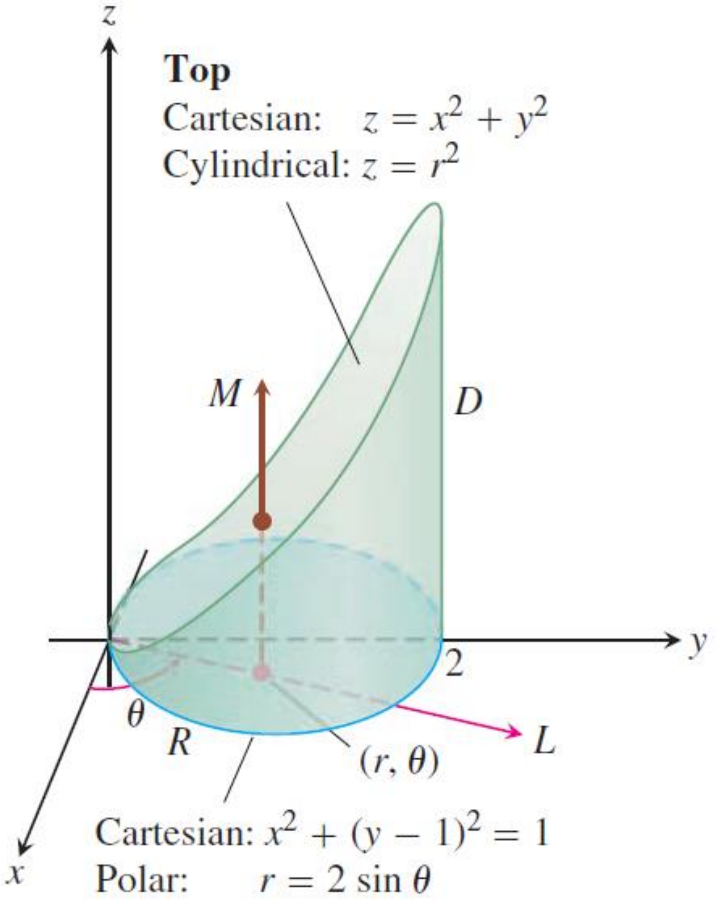
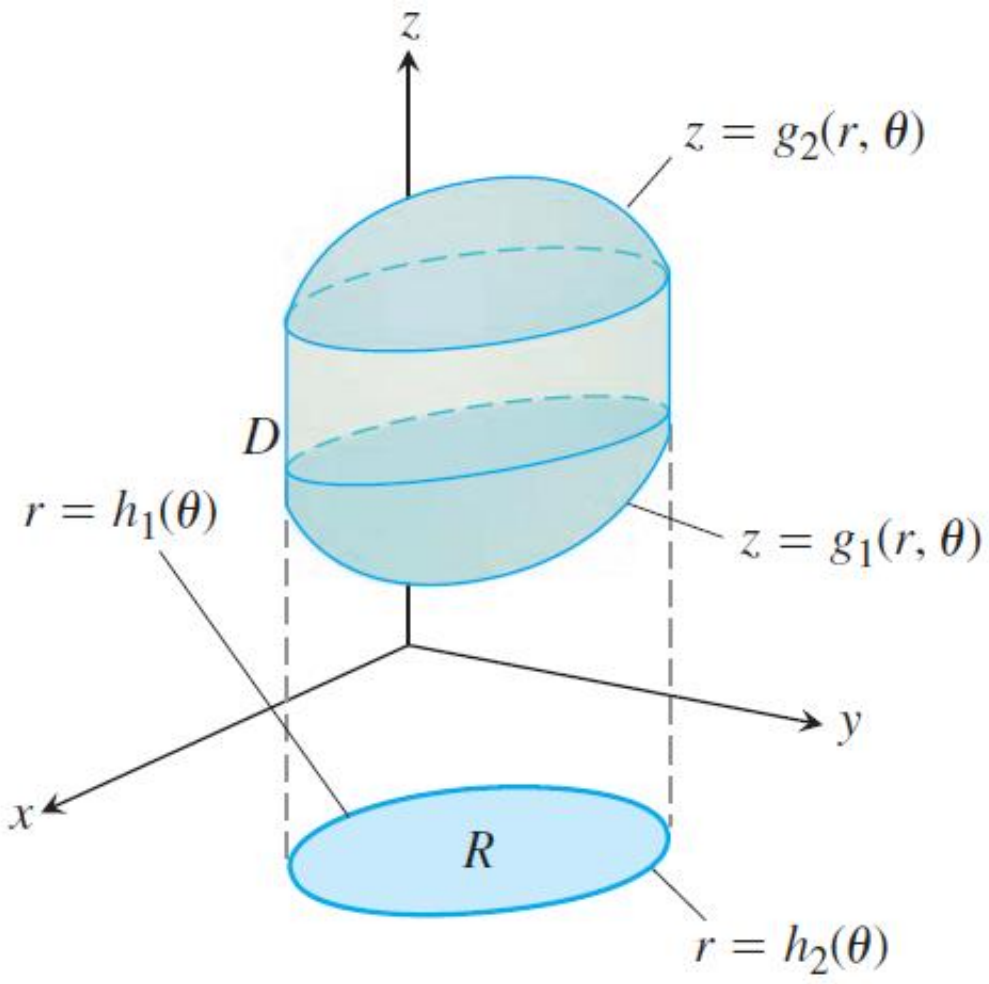


FIGURE 15.44 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

EXAMPLE 1 Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.



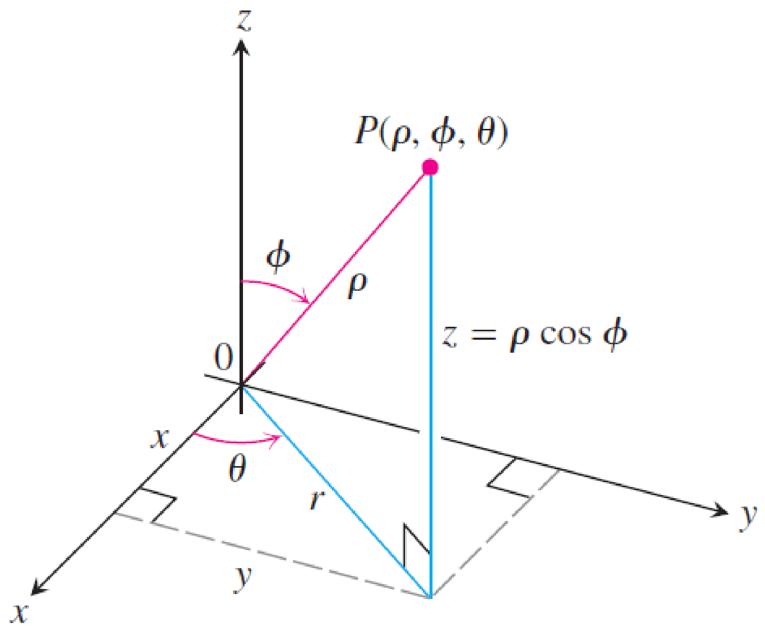
How to Integrate in Cylindrical Coordinates



Spherical Coordinates and Integration

DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
3. θ is the angle from cylindrical coordinates $(0 \leq \theta \leq 2\pi)$.



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$\begin{aligned} r &= \rho \sin \phi, \qquad & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\ z &= \rho \cos \phi, \qquad & y &= r \sin \theta = \rho \sin \phi \sin \theta, \\ \rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}. \end{aligned} \tag{1}$$

Volume Differential in Spherical Coordinates

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

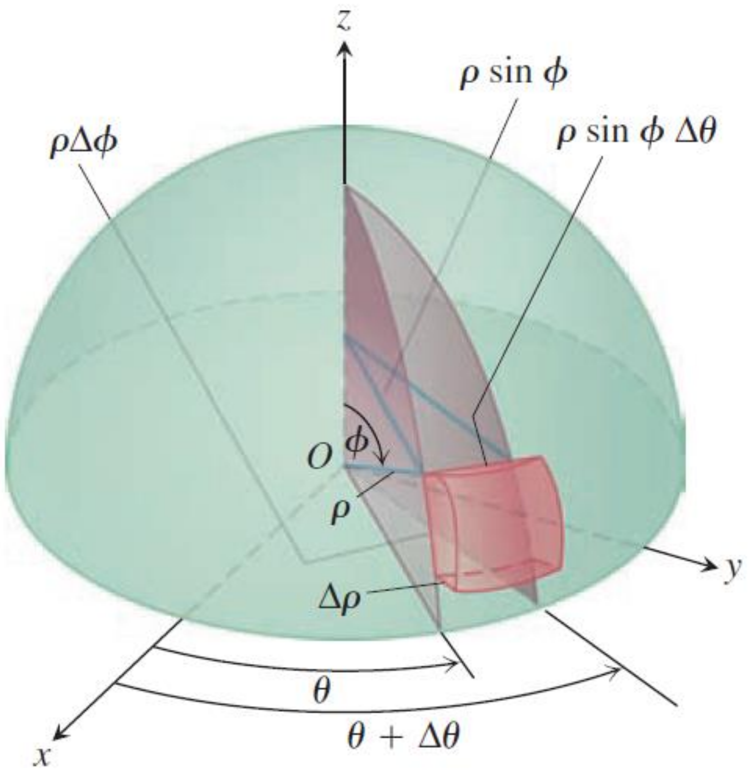


FIGURE 15.51 In spherical coordinates

$$dV = d\rho \cdot \rho \, d\phi \cdot \rho \sin\phi \, d\theta = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$\iiint_D f(\rho, \phi, \theta) \, dV = \iiint_D f(\rho, \phi, \theta) \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$